University College London
Department of Physics and Astronomy
2246E Mathematical Methods III
Coursework M3 (2007-2008)

Solutions to be handed in on Wednesday, December, 12th, 2007

1. The Legendre polynomials satisfy the recurrence relations

$$\begin{aligned}(2n+1)xP_n(x) &= (n+1)P_{n+1}(x) + nP_{n-1}(x) \ , \\ (2n+1)P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x) \ ,\end{aligned}$$

and are normalised such that

$$\int_{-1}^{+1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{nm} \ .$$

Use these results to evaluate

$$\int_{-1}^{+1} P_{n+1}(x)xP_n(x)dx$$

3 MARKS

and show that

$$\int_{-1}^{+1} P'_{n+1}(x)P_n(x)dx = 2 \ .$$

3 MARKS

Using the Legendre polynomials from the lecture notes, verify both relations by explicitly integration for the case of $n = 1$. 4 MARKS

2. The odd function f(x) has the following properties

   (a) $f(x) = -f(-x)$;
   (b) $f(x) = f(x + 2\pi)$;
   (c) $f(x) = \sin(\frac{1}{2}x)$, for $-\pi < x < +\pi$.
   (d) $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$,
   where the Fourier coefficients are

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx \ .$$

Turn sheet over →

Obtain the coefficients b_n and show that the Fourier series is

$$f(x) = \frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{n}{1-4n^2}\sin nx \ .$$

6 MARKS

Evaluate the Fourier series at $x = \pi$ and comment on the answer. 2 MARKS

State Parseval's theorem and use it to evaluate

$$\sum_{n=1}^{\infty} \frac{n^2}{(4n^2-1)^2} \ .$$

6 MARKS

3. The Hermite polynomials $H_n(x)$ arise in the solution of the one-dimensional simple harmonic oscillator in quantum mechanics. They may be defined by the generating function

$$g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)\, t^n \ .$$

By differentiating both sides either with respect to x or t, derive the recurrence relations

$$\frac{d}{dx} H_n(x) = 2n\, H_{n-1}(x) \ ,$$

3 MARKS

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x) \ .$$

4 MARKS

Given that $H_0(x) = 1$, and $H_1(x) = 2x$, derive the form of $H_2(x)$ from the second recurrence relation and show that it satisfies the first one. 3 MARKS

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Model Answers

1. Solution to Problem 1

$$
\begin{aligned}
(2n+1)xP_n(x) &= (n+1)P_{n+1}(x) + nP_{n-1}(x)\ , && (1)\\
(2n+1)P_n(x) &= P'_{n+1}(x) - P'_{n-1}(x)\ , && (2)\\
\int_{-1}^{+1} P_n(x)P_m(x)dx &= \frac{2}{2n+1}\delta_{nm}\ . && (3)
\end{aligned}
$$

$$\int_{-1}^{+1} P_{n+1}(x)xP_n(x)\,dx =$$

from eqn. 1

$$
\begin{aligned}
&= \int_{-1}^{+1} P_{n+1}(x)\left[\frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x)\right]dx\\
&= \frac{n+1}{2n+1}\frac{2}{2(n+1)+1}\\
&= \frac{n+1}{2n+1}\frac{2}{2n+3}
\end{aligned}
$$

$$\int_{-1}^{+1} P'_{n+1}(x)P_n(x)\,dx =$$

NB: From eqn. 2

$$P'_{n+1}(x) = (2n+1)P_n(x) + P'_{n-1}(x)$$

hence

$$
\begin{aligned}
&= \int_{-1}^{+1}\left[(2n+1)P_n(x) + P'_{n-1}(x)\right]P_n(x)\,dx\\
&= (2n+1)\frac{2}{2n+1} + \int_{-1}^{+1} P'_{n-1}(x)P_n(x)\,dx\\
&= 2 + \int_{-1}^{+1} P_{n-3}(x)P_n(x)\,dx = \ldots = \begin{cases} 2 + \int_{-1}^{+1} P'_1(x)P_n(x)\,dx & n \text{ even},\ n \neq 0\\ 2 + \int_{-1}^{+1} P'_0(x)P_n(x)\,dx & n \text{ odd}\end{cases}
\end{aligned}
$$

Since $P_0' = 0$ the n odd case is '2' and $P_1' = 1$ for n even we have to solve the integral

$$\int_{-1}^{+1} P_1'(x)P_n(x)\,dx = \int_{-1}^{+1} 1 \cdot P_n(x)\,dx = \int_{-1}^{+1} P_0(x)P_n(x)\,dx = 0$$

where the last equality follow from eqn. 3 for $n \neq 0$. So overall we obtain

$$\int_{-1}^{+1} P_{n+1}'(x)P_n(x)\,dx = 2$$

For $n = 1$:

$$\int_{-1}^{+1} P_2(x)xP_1(x)\,dx = \int_{-1}^{+1} \frac{1}{2}\left(3x^2 - 1\right)x^2\,dx = \frac{4}{15} = \frac{2 \cdot 2}{3 \cdot 5} \surd$$

$P_2'(x) = 3x$, hence

$$\int_{-1}^{+1} 3x \cdot x\,dx = 2 \surd$$

2. Solution to Problem 2

$$b_n = \frac{2}{\pi}\int_0^{\pi} \sin\frac{1}{2}x\,\sin nx\,dx$$

We solve the integral by integrating by parts twice

$$u' = \sin\frac{1}{2}x \qquad\qquad v = \sin nx$$

$$u = -2\cos\frac{1}{2}x \qquad\qquad v' = n\cos nx$$

$$I_n = \int_0^{\pi} \sin\frac{1}{2}x\,\sin nx\,dx = -2\cos\frac{1}{2}x\sin nx\Big|_0^{\pi} + 2n\int_0^{\pi}\cos\frac{1}{2}x\,\cos nx\,dx$$

$$I_n = 2n\int_0^{\pi}\cos\frac{1}{2}x\,\cos nx\,dx$$

$$u' = \cos\frac{1}{2}x \qquad\qquad v = \cos nx$$

$$u = 2\sin\frac{1}{2}x \qquad\qquad v' = -n\sin nx$$

$$I_n = 2n\left[2\sin\frac{1}{2}x\cos nx\Big|_0^{\pi} + 2n\int_0^{\pi}\sin\frac{1}{2}x\,\sin nx\,dx\right]$$

$$I_n = 4(-1)^n + 4n^2 I_n$$

hence

$$I_n = \frac{4n(-1)^n}{1-4n^2}$$

and

$$b_n = \frac{8}{\pi}\frac{n(-1)^n}{1-4n^2}$$

For $x = \pi$ the function is discontinous and jumps from+1 to -1. The Fourier series results in

$$\frac{8}{\pi}\sum_{n=1}^{\infty}(-1)^n \frac{n}{1-4n^2}\cdot 0 = 0$$

hence the Fourier serious results in the mean: $\frac{1}{2}\lim_{\varepsilon\to 0}\left[f(\pi-\varepsilon)+f(\pi+\varepsilon)\right]$.
Parseval's equality

$$\left\langle f^2(x)\right\rangle = \frac{a_0^2}{2} + \frac{1}{2}\sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)$$

Here:

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\sin^2\frac{1}{2}x\,dx = \frac{32}{\pi^2}\sum_{n=1}^{\infty}\frac{n^2}{(1-4n^2)^2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{+\pi}\sin^2\frac{1}{2}x\,dx = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\frac{1-\cos x}{2}\,dx = \frac{1}{2\pi}\left[\frac{1}{2}x - \frac{1}{2}\sin x\Big|_{-\pi}^{+\pi}\right] = \frac{1}{2}$$

and hence

$$\sum_{n=1}^{\infty}\frac{n^2}{(1-4n^2)^2} = \frac{\pi^2}{64}$$

3. Solutions to Problem 3.

$$g(x,t) = e^{2xt-t^2} = \sum_{n=0}^{\infty}\frac{1}{n!}H_n(x)t^n$$

$$\frac{\partial g}{\partial x} = 2tg = \sum_{n=0}^{\infty}H_n'(x)t^n$$

$$2t\sum_{n=0}^{\infty}H_n(x)t^n = \sum_{n=0}^{\infty}H_n'(x)t^n$$

$$
\begin{aligned}
2\sum_{n=0}^{\infty} H_n(x)t^{n+1} &= \sum_{n=0}^{\infty} H_n'(x)t^n \\
2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_{n-1}(x)t^n &= \sum_{n=0}^{\infty} H_n'(x)t^n
\end{aligned}
$$

Hence we obtain for $n \geq 1$:

$$
\frac{2}{(n-1)!)} H_{n-1}(x) = \frac{1}{n!} H_n'(x)
$$

and

$$
2nH_{n-1}(x) = H_n'(x)
$$

Also we have:

$$
\begin{aligned}
\frac{\partial g}{\partial t} = (2x - 2t)g &= \sum \frac{n}{n!} H_n(x)t^{n-1} \\
2(x-t)g &= \sum_{n-1}^{\infty} \frac{1}{n!} H_{n+1}t^n \\
2x\sum_{n=0}^{\infty} \frac{1}{n!} H_n(x)t^n - 2\sum_{n=1}^{\infty} \frac{1}{(n-1)!} H_{n-1}(x)t^n &= \sum_{n=-1} \frac{1}{n!} H_{n+1}(x)t^n
\end{aligned}
$$

hence for $n \geq 1$:

$$
\frac{2x}{n!} H_n(x) - 2H_{n-1}(x)\frac{1}{(n-1)!} = \frac{1}{n!} H_{n+1}(x)
$$

and then

$$
2xH_n(x) - 2nH_{n-1}(x) = H_{n+1}(x)
$$

With $H_0(x) = 1$ and $H_1(x) = 2x$ we obtain

$$
H_2(x) = 4x^2 - 2
$$