Topic 24 — Diffraction

L24.1 Diffraction by an aperture

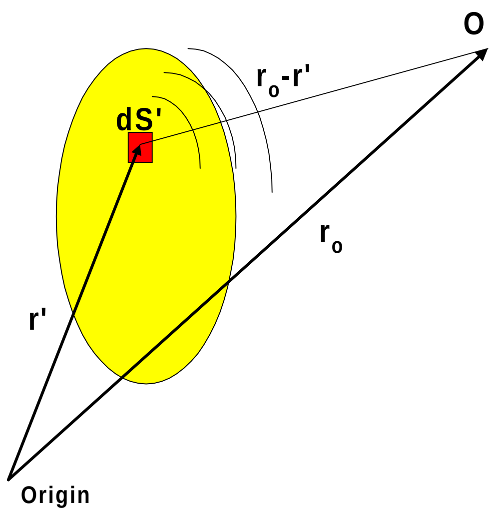

Figure L24.1: Diffraction of a wave through an aperture. Each element dS' of the aperture contributes to the total field at the observation point O.

When a wave is diffracted by an aperture[1], as shown in Figure L24.1, so that the original wave is plane in the aperture, each spherical wave starting from $\mathbf{r}'$ and observed at $\mathbf{r}_o$ is of the form

$$E_n(\mathbf{r}_o, t) = \frac{A}{k|\mathbf{r}_o - \mathbf{r}'|} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)}$$

where the secondary source strength, A, is proportional to the amplitude of the primary wave at $\mathbf{r}'$.

If the primary wave has a constant amplitude over the relevant region, we can write the total field at $\mathbf{r}_o$ as a sum over all the secondary wavelets, or, in the limit, as an integral

$$\begin{aligned} E(\mathbf{r}_o, t) &= \sum \frac{A}{k|\mathbf{r}_o - \mathbf{r}'|} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)} \\ &= \int_{\text{primary wavefront S}} \frac{A(r')}{k|\mathbf{r}_o - \mathbf{r}'|} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)} \mathrm{d}S' \\ &= A \int_{\text{primary wavefront S}} \frac{1}{k|\mathbf{r}_o - \mathbf{r}'|} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)} \mathrm{d}S' \end{aligned}$$

where $\mathbf{r}'$ is a point on the primary wavefront S and $\mathrm{d}S'$ is the area element of S at $\mathbf{r}'$.

Phase and amplitude

One further detail that Fresnel found necessary was to write the constant A_0 as $ikE_0/2\pi$, where E_0 is the magnitude of the electric field on the diffracting

[1]The science of diffraction took a major step forward with the publication in 1665 of the snappily titled *A Physical and Mathematical Thesis on Light, Colours, the Rainbow, and Other Related Topics* by Father Francesco Maria Grimaldi (1618-1663). Grimaldi observed coloured diffraction fringes at the boundaries of shadows using white light. Newton, repeating Grimaldi's experiments, asked "are not the rays of Light in passing by the edges and sides of Bodies, bent several times backward and forwards, with a motion like that of an Eel? And do not the three Fringes of colour'd Light mention'd arise from such bendings?" Newton never got to the bottom of this, and never acknowledged the wave theory of light even though it was available 14 years before he published his *Optiks* in 1704. He did not take up the wave theory in later editions of *Optiks* in 1717 or 1721, or in the 1730 edition "corrected by the Author's own Hand, and left before his Death with the Bookseller." Nevertheless, some of his speculations have (no pun intended) strong wave overtones. For example, in his discussions of the ether he asked "Query 23: is not Vision perform'd chiefly by the Vibrations of this Medium, extended to the bottom of the Eye by the Rays of Light, and propagated through the solid, pellucid and uniform Capillimenta of the Optick Nerves into the place of Sensation....?"

surface. Note that there is a phase change of $\pi/2$ in the expression. We can ignore this detail, as we shall always look at intensities in the end, so this phase information will be lost.

That gives us a recipe for computing the field at any point as a result of a known primary wave.

Kirchhoff's correction

The second problem is that there no backward-going wavelets – why does the wave keep moving forwards? The basic answer is that that if we do the maths properly (Kirchhoff) we find an extra factor in the integral involving the angles of the source-primary wave surface vector and the primary wave-surface-observation point vector with the wave surface — if the source is in line with the centre of the aperture this *obliquity factor* works out to be

$$K(\theta) = \frac{1}{2}(1 + \cos(\theta))$$

where θ is the angle of the outgoing wave from the normal to the primary wavefront.

As long as we avoid large angles, we can take $K(\theta)$ to be 1, and the approximation

$$E(\mathbf{r}_o, t) = A \int_{\text{primary wavefront S}} \frac{1}{k|\mathbf{r}_o - \mathbf{r}'|} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)} \mathrm{d}S'$$

is adequate as an approximation to an exact solution of the wave equation with the appropriate boundary conditions.

If we ignore the distance dependence of the amplitude,

$$E(\mathbf{r}_o, t) = \text{constant} \int_{\text{primary wavefront S}} e^{i(\omega t - k|\mathbf{r}_o - \mathbf{r}'|)} \mathrm{d}S'$$

or, in compressed notation

$$E = \text{constant } e^{i\omega t} \int_{\text{primary wavefront S}} e^{-ikr} \mathrm{d}S$$

where r is the distance from the primary wavefront to the point of observation.

Diffraction pattern at long range

For simplicity, take the origin of the coordinate system at the centre of the aperture. Then the distance from the point (x', y', z') on the primary wavefront to the observation point Q is $r = |\mathbf{r}_o - \mathbf{r}'|$ so that

$$r^2 = x_o^2 + (y_o - y')^2 + (z_o - z')^2,$$

and if we define D as the distance of the observation point from the centre of the aperture

$$D^2 = x_o^2 + y_o^2 + z_o^2$$

so

$$r = \left(D^2 - 2y'y_o - 2z'z_o + z'^2 + z'^2\right)^{1/2}$$

which is, expanding,

$$r \approx D - \frac{y_o}{D}y' - \frac{z_o}{D}z' + O(y^2, z^2)$$

where the notation $O()$ denotes 'terms of order'. If the aperture is small, we may ignore the terms in y'^2 and z'^2.

Then, ignoring the variation of amplitude with distance, and absorbing the phase change on the path length D into the constant term, we have the comparatively simple result

$$E = \text{constant} \int_S e^{i(ky'y_o/D + kz'z_o/D)} \mathrm{d}S'.$$

[2]

[2] A crucial point is that if the integration over S can be evaluated in Cartesian coordinates, the integral can be factorized. For brevity, write $\alpha = \frac{y_o}{D}$ and $\beta = \frac{z_o}{D}$, so that

$$E = \text{constant} \int_{y_{\min}}^{y_{\max}} e^{ik\alpha y'} \mathrm{d}y' \int_{z_{\min}}^{z_{\max}} e^{ik\beta z')} \mathrm{d}z'$$

and if we remember that in the spirit of the Fraunhofer limit we are treating the waves as if they were plane, so that we may write

$$k\alpha = k_y$$

and

$$k\beta = k_z$$

so that

$$E = \text{constant} \int_{y_{\min}}^{y_{\max}} e^{ik_y y'} \mathrm{d}y' \int_{z_{\min}}^{z_{\max}} e^{ik_z z')} \mathrm{d}z'.$$

slit *AF939*

To see how this works out in practice, let us look first at the diffraction pattern of a single slit. If we observe at an angle θ, and the slit is parallel to the z axis and very long, the diffraction pattern will depend only on y_o/D. If we observe at an angle θ to the normal, $y_o/D = \sin(\theta)$. Then we may write, for a slit of width d,

$$\begin{aligned} E &= \text{constant} \int_{-d/2}^{d/2} e^{ik\sin(\theta)y'} \mathrm{d}y' \\ &= \frac{\text{constant}}{ik\sin(\theta)}(2i)\sin(kd\sin(\theta)/2) \end{aligned}$$

That is, the intensity varies as

$$I(\theta) = I(0)\left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{kd}{2}\sin(\theta)}\right]^2.$$

The resulting intensity pattern is shown graphically in figure L24.2, and as a pattern of light and dark bands in figure L24.3.

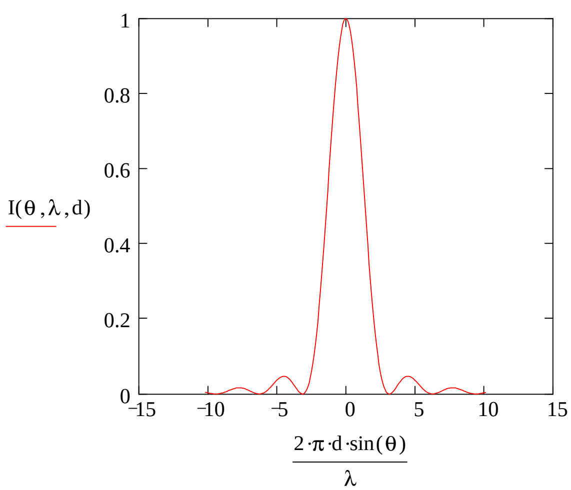

Figure L24.2: The variation of intensity with angle in the diffraction pattern of a narrow slit.

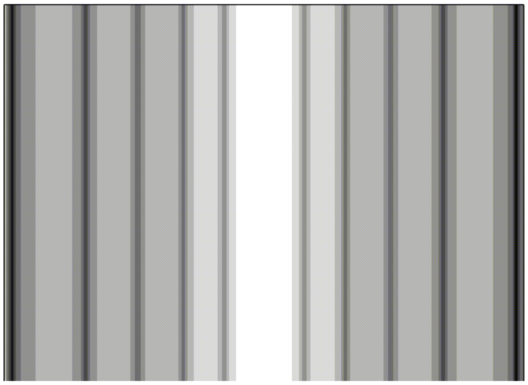

Figure L24.3: The light and dark bands in the diffraction pattern of a narrow slit.

The intensity will be zero whenever the argument of the sin function in the numerator is a multiple of π, that is when

$$\frac{kd}{2}\sin(\theta) = m\pi$$

where m is an integer. The first zero of intensity is at

$$\frac{kd\sin(\theta)}{2} = \pi$$

or, as the angle θ will be small,

$$\theta = 2\pi/kd = \lambda/d.$$

Note that the central peak is the highest, and broader than the subsidiary peaks. The subsidiary peaks decrease in amplitude with distance away from the centre.

The successive peaks (different m) are known as different *orders* of diffraction.