Topic 17 — - Waves in more than one dimension *FGT377, AF769-772*

The waves we have discussed so far have been propagating only along the x-axis. However, we have seen that the *sign* of the kx term determines whether the wave is propagating in the positive or negative direction. This gives us a clue that k somehow has a direction associated with it - and so it does. When we come to consider waves in more than one dimension we see that k should be treated as a vector.

plane waves in three dimensions

Figure T17.1: Planes of constant phase in a plane wave in three dimensions are parallel planes (figure omitted to save file space).

First, let's ask what the waves that we have been dealing with so far look like. We have had, for example for sound waves in air, a displacement ξ which depended only on x. Thus a particular phase (corresponding to a particular value of $\omega t - kx$) exists over a whole plane perpendicular to the x axis - we have *plane phase fronts* or *plane wave fronts*. Was there anything special about the x axis? No, we could pick any direction (see figure T17.1. Our maths tells us, though, that $\mathbf{k}.\mathbf{r} = constant$ where $\mathbf{r} = (x, y, z)$ defines a plane in three dimensional space, perpendicular to $\mathbf{k}$.

We can therefore generalize our expression for a wave to three dimensions, as

$$\xi(\mathbf{r}, t) = Ae^{i(\omega t - \mathbf{k}.\mathbf{r})} \tag{T17.1}$$

for a wave travelling in the *direction* $\mathbf{k}/|\mathbf{k}|$ and with a wavelength $2\pi/|\mathbf{k}|$.

The phase velocity of the wave is, as before, $v_{\mathrm{p}} = \omega/|\mathbf{k}|$. Sometimes the *direction* of propagation of the wave, the unit vector $\mathbf{k}/|\mathbf{k}|$ is written $\hat{\mathbf{k}}$.

For example, if $\mathbf{k} = k(1/\sqrt{2}, 1/\sqrt{2}, 0)$ we have a wave travelling at 45 degrees to the x axis. If $\mathbf{k} = k(0, 0, -1)$ we have a wave travelling down the z axis towards $z = -\infty$.

The wave equation, too, is generalised in more than one dimension:

$$\frac{\partial^2 \xi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2} \Rightarrow \nabla^2 \xi = \frac{1}{c^2}\frac{\partial^2 \xi}{\partial t^2}. \tag{T17.2}$$

The Laplacian, ∇^2, has a particularly simple form in normal Cartesian coordinates:

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2},$$

so that if we write

$$\mathbf{k} = (k_x, k_y, k_z),$$

it follows, by substituting equation T17.1 into equation T17.2, that

$$k_x^2 + k_y^2 + k_z^2 = \frac{\omega^2}{c^2},$$

which we may write, as $k_x^2 + k_y^2 + k_z^2 = |\mathbf{k}|^2$ as

$$|\mathbf{k}| = \frac{\omega}{c}.$$

Often we will write $|\mathbf{k}|$ as k.

cylindrical and spherical waves — energy fluxes

In practice, of course, we cannot generate plane waves with infinite, flat wavefronts - to do so would require infinitely large, flat sources. Let us look at the other extreme, when our sources are lines or even points. The first point to worry about is the conservation of energy. We know that in a plane wave we can write the intensity, that is the energy passing unit area in unit time, of a plane wave of amplitude ξ_0

$$I = \frac{1}{2} Z \omega^2 \xi_0^2$$

where Z is a constant, the specific impedance, characteristic of the medium, and ω the angular frequency.

Suppose we have a cylindrical source. What does this mean? It means the surfaces of constant phase (the wave fronts) are cylindrical. Think of the ripples on a pond when you drop in a stone - they spread as ever-increasing circles. Now the total rate at which energy passes through a cylinder of radius r around the source will be

$$2\pi r I$$

which will increase with r if I is constant - which violates the conservation of energy - the only energy input is at our source. If the energy flow is to be independent of r, we must have

$$I \approx \frac{1}{r},$$

or

$$\xi \approx \frac{1}{\sqrt{r}}.$$

The amplitude dies off with distance from the source. This applies equally to a line source in three dimensions (perhaps a very long strip light) or a point source in two dimensions (a drop falling onto the surface of a liquid).

The same happens in three dimensions, expect that the rate at which energy passes through a sphere of radius r is

$$4\pi r^2 I$$

so that in three dimensions with a point source we have an inverse square law for intensity[1] and

$$\xi \approx \frac{1}{r}.$$

How would we write a spherical wave? We want constant phase at a constant value of r, so a suitable form is

$$\xi(r,t) = \frac{1}{kr} e^{i(\omega t - kr)}.$$

This is independent of the other spherical coordinates θ and ϕ[2].

T17.1 Curvature of Wavefront

The curvature of the wavefront from a point source decreases with distance from the source (see figure T17.2). In many situations, we are far enough

[1] The inverse square law was probably first stated by Johannes Kepler (1571-1630) in his book on optics of 1604.

[2] You may ask why this appears not to be of the form $f(ct \pm x)$. The point is that when we transform from Cartesian (x, y, z) coordinates to spherical polar (r, θ, ϕ) the second spatial derivative for r becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}$$

and applying this to

$$\frac{1}{kr} e^{i(\omega t - kr)}$$

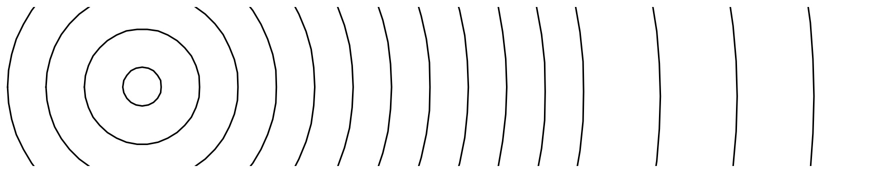

Figure T17.2: A wave spreading away from a point source, showing that at large distances a small section of the wavefront looks almost flat.

away from the source that we may neglect the curvature of the wavefront, and treat the wave as locally plane.

If that is the case, what do we do about

- the variation of amplitude with distance from the source;
- the direction of travel of the wave?

We ignore the first, because if we are looking at phenomena in which path differences (i.e. the different values of r) are small compared with r the changes in amplitude are small compared with the overall swings from + to − as a result of the wave's oscillations. Consider, for example, how the wave field will alter as we move from, say, 100 wavelengths from the source to 101 wavelengths away. In that distance the change in amplitude of a spherical wave, as a result of the $1/r$ factor, will be only 1 percent, but the wave will have changed phase by 2π, and so it will have covered all the values between (plus the amplitude) and (minus the amplitude).

---

gives

$$\frac{1}{r^2}\frac{\partial}{\partial r}r^2\left(\frac{-ik}{kr}e^{i(\omega t-kr)}-\frac{1}{kr^2}e^{i(\omega t-kr)}\right)$$

or

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(-ire^{i(\omega t-kr)}-\frac{1}{k}e^{i(\omega t-kr)}\right)$$

which gives

$$\frac{1}{r^2}\left(-ie^{i(\omega t-kr)}+i^2kre^{i(\omega t-kr)}-\frac{-ik}{k}e^{i(\omega t-kr)}\right)$$

or

$$-k^2\frac{1}{kr}e^{i(\omega t-kr)}.$$

So our expression will satisfy the wave equation.

For the second, we know where the wave came from, i.e. we know the direction of $\mathbf{k}$, so if the source is at $\mathbf{r}_0$ and we observe the wave at $\mathbf{r}$ the locally plane piece of wavefront we see is travelling in the direction $(\widehat{\mathbf{r} - \mathbf{r}_0})$, where the hat denotes a unit vector. Thus we can "convert" the spherical wave to a plane wave of the form

$$e^{i(\omega t - k\ (\widehat{\mathbf{r}-\mathbf{r}_0}).\mathbf{r})}.$$