Topic 14 — Reflection and Transmission at Interfaces (concluded)

numerical example

Consider a join between two strings, one of which has four times the mass per unit length of the other. Thus the impedance of the second is double that of the first, and we have

$$\mathtt{r} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{1-2}{1+2} = -\frac{1}{3}$$

and

$$\mathtt{t} = \frac{2Z_1}{Z_1 + Z_2} = \frac{2 \times 1}{1+2} = \frac{2}{3}.$$

Figure T14.1 shows a snapshot of the wave: we can see here that there is a change of wavelength between the two halves of the string, and that the transmitted amplitude is 2/3. We can also see how the incident and reflected waves on the left combine, with their amplitudes of 1 and -1/3, to match the displacement at the join. We can zoom in on the join, and see that it is indeed smooth.

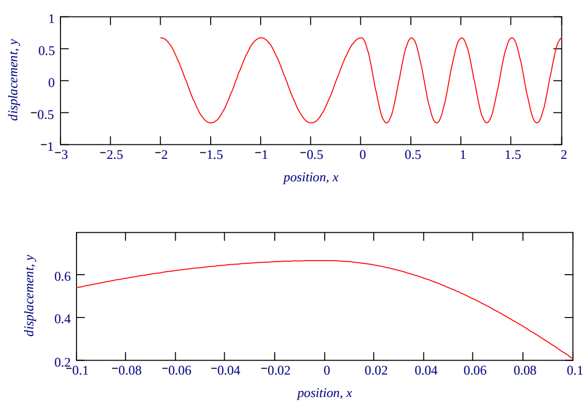

Figure T14.1: A wave encountering the join between two strings: the lower figure is a close-up near the join, showing the smoothness of the displacement.

If we look at the first derivative with respect to x, which is proportional to the restoring force, we see that this is also continuous (upper part of figure T14.2), although it has a change in slope at the join. The second derivative, however (lower part of figure T14.2), is discontinuous. This must be so, because when we think of the differential equation

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$$

the left-hand side must be continuous at the join (the displacement is continuous, and each successive differentiation with respect to time merely produces an extra factor of $i\omega$, so velocity and acceleration must also be continuous across the join). As a result, the discontinuity in wave speed c must be compensated by a discontinuity in $\partial^2\xi/\partial x^2$.

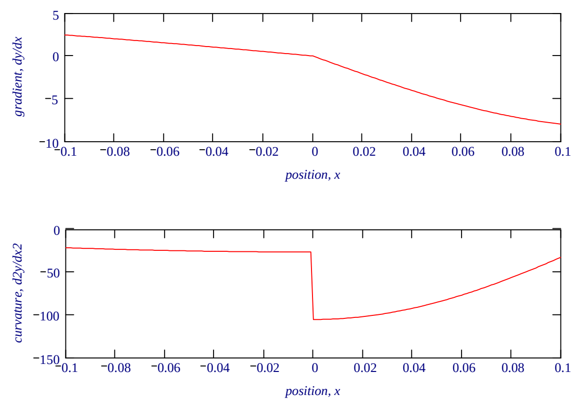

Figure T14.2: The slope and curvature of the wave near the join.

limiting cases of incident, reflected and transmitted waves.

How does these formulae work out for a string with a fixed end? If we take Z_2 to be infinite, which would correspond to an infinitely heavy string which would not move, we find $\mathtt{r} = -1$ and $\mathtt{t} = 0$ – which is just the combination of a positive and negative-going wave of equal amplitude we saw when we looked at standing waves.

What happens if we have a free end, $Z_2 = 0$? Then $\mathtt{r} = 1$, so the displacement of the end is *double* the incident amplitude – which is how whips work.

T14.1 Acoustic waves in dissimilar media - reflection and transmission

a) directly

The quantities which must be continuous at a boundary are the particle displacement (or, equivalently as the frequency is a constant, the particle velocity) and the acoustic excess pressure (the sound pressure). As we've already done one case, we'll canter through this one quite rapidly.

Again, we have incident, reflected and transmitted waves. From continuity of displacement we have

$$1 + \mathtt{r} = \mathtt{t}$$

exactly as before. The pressure is $-B_\mathrm{a}\partial\xi/\partial x$, so the pressure continuity gives us

$$B_{\mathrm{a}1}k_1 - \mathtt{r}B_{\mathrm{a}1}k_1 = \mathtt{t}B_{\mathrm{a}2}k_2.$$

The algebra is just the same as before, and gives

$$\mathtt{r} = \frac{B_{\mathrm{a}1}k_1 - B_{\mathrm{a}2}k_2}{B_{\mathrm{a}1}k_2 + B_{\mathrm{a}2}k_2}$$

But Z_i is $B_{\mathrm{a}i}k_i/\omega$ so

$$\mathtt{r} = \frac{Z_1 - Z_2}{Z_1 + Z_2}$$

as before.

Note: if $Z_2 >> Z_1$ (a hard wall) $\mathtt{r} = -1$, so the total displacement at the wall is zero. The wall will be a displacement *node* – reasonable enough, as we can't move the very unyielding wall – but as displacement and pressure are in antiphase, this gives us a pressure *antinode*.

energy conservation

In many cases we would like to know the *energy* reflection coefficients rather than the *amplitude* coefficents. Provided that the Zs are real (loss-free media) then cancelling the factors of $\frac{1}{2}\omega^2$ we have

$$R = \frac{I_r}{I_i} = \frac{Z_1\xi_r^2}{Z_1\xi_i^2} = \frac{Z_1\mathtt{r}^2}{Z_1 1^2} = \left(\frac{Z_1 - Z_2}{Z_1 + Z_2}\right)^2$$

and

$$T = \frac{I_t}{I_i} = \frac{Z_2 \xi_t^2}{Z_1 \xi_i^2} = \frac{Z_2 \mathtt{t}^2}{Z_1 1^2} = \frac{Z_2}{Z_1}\left(\frac{2Z_1}{Z_1+Z_2}\right)^2 = \frac{4Z_1Z_2}{(Z_1+Z_2)^2}.$$

Note that these values of R and T give

$$R + T = 1, \tag{T14.1}$$

which expresses the conservation of energy.

b) in terms of impedance

There is a pattern emerging here - we get similar expressions for different systems when we express them in terms of impedances. This is a very powerful result, which extends beyond mechanical waves and into electromagnetic waves in which case the impedance is the ratio E/H of the electric field to the magnetic field in the wave.

The reason that this works is that the continuity equations at the boundary between two media can be expressed in terms of the 'force' and 'response' terms of impedance. Continuity of displacement in mechanical case implies continuity of velocity (because the two are related by a constant factor of $i\omega$), i.e. of 'response'. 'Force' (pressure in the acoustic wave, transverse force for the wave on the string) is Z times 'response'. The algebra is then the same as before, giving the expressions for r and t directly in terms of Z.

There is one tricky point, however. In all our mechanical cases we have taken wave amplitudes to be defined by the *displacements*. We could have taken, for example, pressure instead of displacement amplitude — but then the reflection coefficients would have to be rewritten with $1/Z_i$ wherever our present expressions have Z_i (note that this is consistent with, for example, a rigid boundary being a displacement node but a pressure antinode. The reflection coefficient for pressure is the same as for displacement, but with the opposite sign. The transmission amplitude for pressure, however, differs from that for displacement by a factor of Z_2/Z_1). Unfortunately, electromagnetic waves are conventionally described in terms of their electric field amplitude ('force') rather than magnetic field ('response'), and we need to recall this when using formulae for reflection and transmission. The rule is simple, though: for electromagnetic waves, use the formulae we have derived but with refractive index n in place of impedance Z.

Wave system	Impedance
Transverse wave on string	$F/v = \rho c = \sqrt{T\mu}$
Pressure wave in fluid	$p/\dot{\xi} = \rho c = \sqrt{B\rho}$
Elastic wave on rod	$\sigma/\dot{\xi} = \rho c = \sqrt{Y\rho}$
Electromagnetic wave	$E/H = \sqrt{\mu_r\mu_0/\epsilon_r\epsilon_0}$
	Usually $\mu_r = 1$, and so $Z = Z_0/\sqrt{\epsilon_r} = Z_0/n$
	F: force; v: velocity; T: tension; μ: mass per unit length p: pressure; $\dot{\xi}$: velocity; ρ: density; c wave speed; B: bulk modulus σ: stress; Y: Young's modulus μ_r: relative permeability; ϵ_r: relative permittivity (dielectric constant) μ_0: permeability of free space; ϵ_0: permittivity of free space Z_0: impedance of free space; n: refractive index

Table T14.1: Impedances for several wave systems.

Table T14.1 shows a few examples of wave systems and the corresponding impedances.