Topic 13 - Impedance and Energy Transport in Waves (concluded) — Reflection and Transmission at Interfaces

Rate of energy propagation in a gas

Let us prove the result which we produced by a slightly hand-waving argument for the energy propagation in a gas. The rate of energy transfer is the rate at which an element of gas at x does work on a neighbouring element. The pressure is

$$p = -B_{\rm a}\frac{\partial \xi}{\partial x}$$

and, as the rate at which a force does work is the product of the force and the rate of movement in the direction of the force, this does work at a rate $p\partial\xi/\partial t$ per unit area, or the rate of energy transfer per unit area is

$$W(x,t) = -B_{\rm a}\frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial t} = B_{\rm a}\xi_0^2 k\omega sin^2(\omega t - kx),$$

and averaging over a period we have

$$I = \langle W \rangle = \frac{1}{2}B_{\rm a}k\omega\xi_0^2 = \frac{1}{2}\rho_0\omega^2\xi_0^2 c = \frac{1}{2}\omega^2 Z\xi_0^2,$$

the intensity, which is exactly $c\langle E\rangle$.

Measurement of Sound

The loudness of a sound is usually measured on a logarithmic scale - given the number of powers of 10 over which the pressure amplitude (proportional to the square root of the power) varies, this is understandable. A standard of intensity is selected:

$$I_0 = 10^{-12}\ \text{W m}^{-2}$$

which is about the limit of audibility for a human. The typical level of a conversation between two people standing close together is about 10^{-6} W m^{-2}. Other intensities are measured, relative to the standard intensity, in bels.

If $I/I_0 = 10^b$, then I is said to be b bels louder than I_0, or $\log_{10}(I/I_0) = b$.

The bel, a factor of 10, is quite a coarse measure, so the decibel (db) is more frequently. A decibel, of course, is one tenth of a bel.

Correspondingly, one decibel (db) is a factor of $10^{0.1} \approx 1.3$, three decibels (3 db) is $10^{0.3} \approx 2$.

The ear has an intensity range, from just hearing to pain, of about 12 bels. A number of examples are shown in figure T13.1.

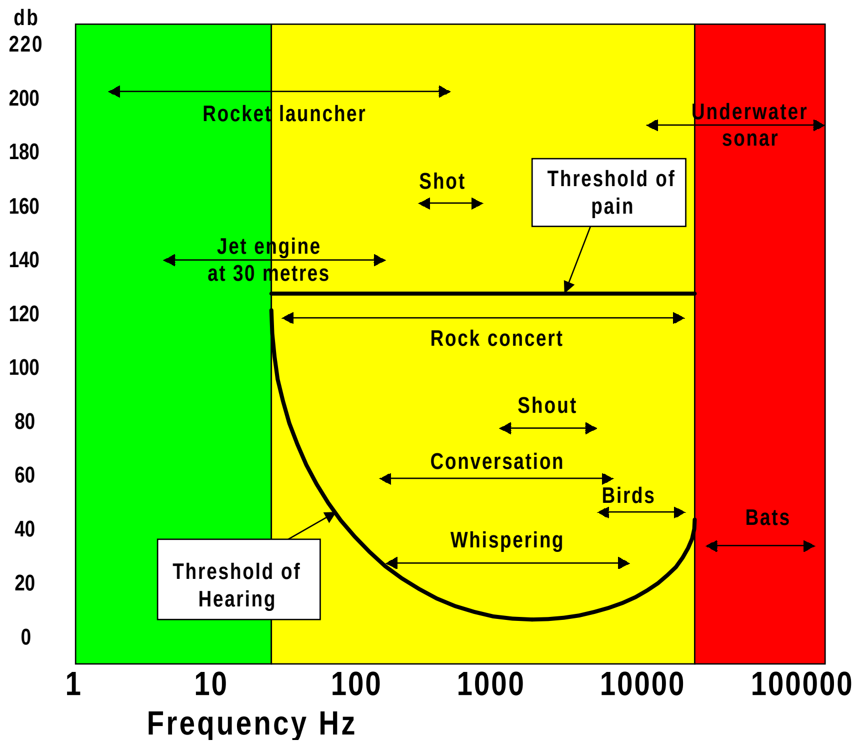

Figure T13.1: Examples of sound intensities.

Note that as well as giving an intensity to a standard, the decibel scale can be used to *compare* two signals: one signal of intensity I_2 is n db stronger than another of intensity I_1 if

$$n = 10 \log_{10} \left(\frac{I_2}{I_1} \right).$$

T13.1 Joined strings *AF863-865*

We've looked at waves on a finite string, and seen the normal modes, which are evidence for waves being reflected at each end. The end of the string is just an example, albeit an extreme one, of a discontinuity in the string – another way of thinking of the rigid support is as an infinitely massive string. Let's now ask what happens when we join two dissimilar strings together. Intuitively we can see that if we tie a more massive string on the end of a light one, it's going to be hard for the light string to shake it hard enough to get all the energy into it. We want to quantify this effect.

Imagine our string, under tension T, running from minus infinity to plus infinity, but with the section from minus infinity to 0 having density ρ_1, that from 0 to infinity having density ρ_2. The tension is the same in the two halves, but the difference in density means that the wave speeds will be different.

What must happen at the boundary? Clearly the displacement of the string must be continuous –otherwise it's broken. Also the transverse force must be continuous – otherwise we have a finite force acting on an infinitesimal element of string, and we will break it.

We send a wave in from minus infinity: let it have unit amplitude

$$y_{\mathrm{i}}(x,t) = e^{i(\omega t - k_1 x)}.$$

We expect some of the wave to be transmitted into the second half

$$y_{\mathrm{t}}(x,t) = \mathtt{t} e^{i(\omega t - k_2 x)},$$

and some to be reflected

$$y_{\mathrm{r}}(x,t) = \mathtt{r} e^{i(\omega t + k_1 x)}.$$

Note that there is no negative-going wave in the second half - anything that gets into that half disappears towards infinity. We are using different typefaces to distinguish the transmitted amplitude $\mathtt{t}$ from the time t.

Our two continuity conditions give

$$\begin{aligned} y_{\mathrm{i}}(0,t) + y_{\mathrm{r}}(0,t) &= y_{\mathrm{t}}(0,t) \\ T\left[\frac{\partial}{\partial x} y_{\mathrm{i}}(0,t) + \frac{\partial}{\partial x} y_{\mathrm{r}}(0,t)\right] &= T\frac{\partial}{\partial x} y_{\mathrm{t}}(0,t). \end{aligned}$$

Substituting, and taking out the common factor of $e^{i\omega t}$, we have

$$1 + \mathtt{r} = \mathtt{t} \qquad \text{(T13.1)}$$

$$-k_1 T + k_1 \mathtt{r} T = -k_2 \mathtt{t} T \qquad \text{(T13.2)}$$

from which we have, substituting for $\mathtt{t}$ from equation T13.1 into equation T13.2,

$$\begin{aligned} -k_1T + k_1\mathtt{r}T + k_2T + k_2\mathtt{r}T &= 0 \\ \mathtt{r} &= \frac{k_1T - k_2T}{k_1T + k_2T} \end{aligned}$$

But all the k_iTs are of the form $\omega T/c_i = \omega Z_i$, and[1] ω is the same throughout, so

$$\boxed{\mathtt{r} = \frac{Z_1 - Z_2}{Z_1 + Z_2}} \qquad \text{(T13.3)}$$

and it follows that

$$\boxed{\mathtt{t} = \frac{2Z_1}{Z_1 + Z_2}.} \qquad \text{(T13.4)}$$

Note that the reflection and transmission coefficients are independent of frequency, that is, independent of wavelength. This is to be expected, as there is no characteristic length in the problem (two semi-infinite strings), so there is no reason to expect a dependence on wavelength.

[1]We have used $T/c = T\sqrt{T/\rho} = \sqrt{T\rho} = Z$.