Topic 12 - Impedance and Energy Transport in Waves

Impedance for sound waves in a gas

We have seen that if we define

$$Z = \text{Specific acoustic impedance} = \frac{\text{excess pressure}}{\text{particle velocity}} = \frac{p}{\dot{\xi}}. \quad \text{(T12.1)}$$

we get

$$Z = \frac{B_a k}{\omega} = \frac{B_a}{c} = \rho_0 c = \sqrt{B_a \rho_0}.$$

This is a constant, which is independent of the frequency of the wave: it is a characteristic of the material itself. This tells us something about how easy it is to get the gas moving, and is thus related to the amount of energy stored in the gas. For the case of the gas which we treated before

$$Z = 1 \times 374 = 374 \text{ kg m}^{-2} \text{ s}^{-1}.$$

Note also that the impedance is *real*[1].

[1] There is a significant difference between the case of a wave and the case of a simple harmonic oscillator.

For water, with a bulk modulus $B = 2.2 \times 10^9$ Pa, density $\rho = 1000$ kg m^{-3}, we have

$$Z = \sqrt{B\rho} = 1.5 \times 10^6 \text{ kg m}^{-2} \text{ s}^{-1},$$

showing that water has a much higher impedance, is much harder to get moving, than air.

- Simple Harmonic Oscillator: real impedance means work is being done on the system. As the energy that is put in cannot escape as a wave, it must be dissipated in the system - in an electric circuit, for example, resistance contributes to the real part of impedance and causes losses, whereas capacitance and impedance have imaginary impedances and with sinusoidal input energy is stored and recovered, not dissipated.
- Wave: real impedance means work is being done on the system, but here the energy is *propagated away* as it is put in. In a wave-bearing system, losses are denoted by an imaginary component to the impedance.

additional example — not done in lecture — of a string *P109-110*

For a wave on a stretched string, the impedance is the ratio of the transverse force to the transverse velocity. Note that here the force is the force tending to accelerate the string, that is the negative of the vertical component of the tension in the string, so if the string's displacement is

$$y(x,t) = Ae^{i(\omega t - kx)}$$

the impedance is

$$Z = \frac{-T\partial y/\partial x}{\partial y/\partial t} = \frac{ikT}{i\omega} = T/c = \rho c,$$

where we have cancelled the common factor of the complex exponential from the top and bottom, and used $\omega/k = c$.

In our previous piano-wire example, then, the impedance will be

$$Z = 0.0125 \times 253 = 3.16 \text{ kg s}^{-1}.$$

T12.1 Energy transport in waves

One of the reasons for spending some time at the start of the course talking about the interchange of energy between different forms in simple harmonic oscillators (e.g. kinetic/potential) was so that we would have a rough idea how to treat energy in waves. A wave represents propagation of energy from one place to another.

Kinetic energy density

As an example, consider the energy in an acoustic wave in a gas. There are two components, kinetic and potential. Again, take a slab of gas $\mathrm{d}x$ thick, with unit cross-sectional area. The kinetic energy, $\frac{1}{2}mv^2$, of the volume $1\mathrm{d}x$ of density ρ moving with velocity $\dot{\xi}$, is

$$E_{\text{kin}}\mathrm{d}x = \frac{1}{2}\rho_0\dot{\xi}^2\mathrm{d}x$$

which varies both in time and space: in fact as the wave passes there will be times when the gas is instantaneously stationary, and in between there will be maxima of kinetic energy. Alternatively, if we look at any instant then there will be points of maximum and of zero kinetic energy. Note that as we have defined it here, $E_{\rm kin}$ is a *kinetic energy density*, the kinetic energy of a unit volume (the volume of our slab was the unit cross-sectional area times the thickness dx. A useful quantity is the average over a wavelength (this is enough, as what happens in one wavelength is repeated exactly in the next wavelength). Remember that if we want an energy we need a real velocity, so

$$\begin{aligned}\dot{\xi} &= \mathrm{Re}\left[\frac{\partial}{\partial t}\xi_0 e^{i(\omega t - kx)}\right] \\ &= -\omega\xi_0\sin(\omega t - kx)\end{aligned}$$

and then the average kinetic energy per unit volume will be

$$\langle E_{\rm kin}\rangle = \frac{\frac{1}{2}\rho_0\omega^2\xi_0^2\int_0^\lambda \sin^2(2\pi(ct-x)/\lambda)\mathrm{d}x}{\lambda} \tag{T12.2}$$

$$= \frac{\frac{1}{2}\rho_0\omega^2\xi_0^2\int_0^\lambda \frac{1}{2}(1-\cos(4\pi(ct-x)/\lambda)\mathrm{d}x}{\lambda} \tag{T12.3}$$

$$= \frac{1}{4}\rho_0\omega^2\xi_0^2 \tag{T12.4}$$

which is very similar in form to the average kinetic energy of a simple harmonic oscillator[2]. Note that the spatial average over a wavelength has also removed the time dependence: we could equally well average the local kinetic energy density over a period of the wave, when we would obtain the same constant value, independent of position x[3].

[2]check the integral yourself before reading on. If you had difficulty, here it is

$$\begin{aligned}\int_0^\lambda \cos(2(\omega t - kx))\mathrm{d}x &= -\frac{1}{2k}\left[\sin(2(\omega t - kx))\right]_0^\lambda \\ &= -\frac{1}{2k}\left[\sin(2(\omega t - k\lambda)) - \sin(2(\omega t))\right] \\ &= -\frac{1}{2k}\left[\sin(2(\omega t - 2\pi)) - \sin(2(\omega t))\right] \\ &= 0\end{aligned}$$

because the wavelength $\lambda = 2\pi/k$ and $\sin(\theta - 4\pi) = \sin(\theta)$ irrespective of the value of θ.

[3]Formally,

$$\int_{t_0}^{t_0+T} \cos(2(\omega t - kx))\mathrm{d}t = \frac{1}{2\omega}\left[\sin(2(\omega t - kx))\right]_{t_0}^{t_0+T}$$

Potential energy density - for interest only

What about the potential energy? The key quantity here is the work done on the gas by the excess pressure in a change of volume from V_0 to $V_0 + v$. We can write this as

$$\begin{aligned}
E_{\text{pot}} &= -\int_{V_0}^{V_0+v} P\mathrm{d}V \\
&= -\int_{V_0}^{V_0+v} (P_0 + p)\mathrm{d}V \\
&= -\int_{V_0}^{V_0+v} \left(P_0 - B_a \frac{V - V_0}{V_0}\right) \mathrm{d}V \\
&= -\left[P_0 V - \frac{1}{2} B_a \frac{V^2}{V_0} + B_a V\right]_{V_0}^{V_0+v} \\
&= -\left[(P_0 + B_a)v - \frac{1}{2} B_a \frac{(V_0+v)^2}{V_0} + \frac{1}{2} B_a \frac{V_0^2}{V_0}\right] \\
&= -\left[P_0 v - \frac{1}{2} B_a \frac{v^2}{V_0}\right] \\
&= -\left[P_0 v + \frac{1}{2} p v\right] \qquad \text{(T12.5)}
\end{aligned}$$

Now we expect the first term in equation T12.5 — it's the one which corresponds in an infinitesimal change of volume to

$$dE_{\text{pot}} = -P_0 \mathrm{d}V : \qquad \text{(T12.6)}$$

the point is that we need to concentrate on the meaning of that volume change $\mathrm{d}V$, which corresponds to v. In Eq T12.6 we are considering an infinitesimal change in volume of a finite volume of gas, so that the volume strain (change in volume per unit volume) is infinitesimal. In the sound wave, by contrast, the volume we are considering is infinitesimal (the slab of thickness $\mathrm{d}x$) but the volume strain $v/V_0 = \partial\xi/\partial x$ is finite, and it is integrating up to this

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$$\begin{aligned}
&= \frac{1}{2\omega}\left[\sin(2(\omega(t_0+T) - kx)) - \sin(2(\omega t_0 - kx))\right] \\
&= \frac{1}{2\omega}\left[\sin(2(\omega t_0 + 2\pi - kx)) - \sin(2(\omega t_0 - kx))\right] \\
&= 0
\end{aligned}$$

because the period $T = 2\pi/\omega$ and $\sin(\theta + 4\pi) = \sin(\theta)$ irrespective of the value of θ.

finite volume strain which gives rise to the factor of one half[4]. But we know from before that

$$p = -B_{\mathrm{a}} \frac{\partial \xi}{\partial x} \tag{T12.7}$$

and the change in volume v is, for unit area,

$$v = \frac{\partial \xi}{\partial x} \mathrm{d}x, \tag{T12.8}$$

where

$$\frac{\partial \xi}{\partial x} = k\xi_0 \sin(\omega t - kx). \tag{T12.9}$$

Now consider the average over a wavelength of the first term in equation T12.5. P_0 is a constant, so we have the integral of a sine over a whole period, which is zero. The pv term, however, gives a $\sin^2$ term (from equations T12.7, T12.8 and T12.9), so, in the same way as before, and dividing by V_0 to get an energy density (energy per volume)

$$\langle E_{\mathrm{pot}} \rangle = \frac{1}{4} B_{\mathrm{a}} k^2 \xi_0^2$$

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[4] An alternative derivation works in terms of the relative compression of the gas, known as the *condensation* $s = -\partial\xi/\partial x$. Consider a small mas of gas, initially with volume V_0, finally with volume V_1. The corresponding values of the condensation will by 0 and s_1, and at some intermediate time the volume and condensation will be V and s. Then each small change in condensation ds corresponds to a change in volume

$$\mathrm{d}V = -V_0 \mathrm{d}s.$$

The total pressure, P, is

$$\begin{aligned} P &= P_0 + p \\ &= P_0 - B_a s, \end{aligned}$$

and so the potential energy is

$$\begin{aligned} E_{\mathrm{pot}} &= \int_0^{s_1} (P_0 - B_a s) V_0 \mathrm{d}s \\ &= P_0 V_0 s_1 + \frac{1}{2} B_a V_0 s_1^2. \end{aligned}$$

Now take the volume V_0 to be the infinitesimal volume (unit cross-section)$\times$dx, and note that when we integrate over a wavelength the first term, linear in s_1, will have cancelling positive and negative regions, leaving only the term involving the square of s_1.

but $k^2 = \omega^2/c^2 = \omega^2\rho_0/B_\mathrm{a}$, so

$$\langle E_\mathrm{pot}\rangle = \frac{1}{4}\rho_0\omega^2\xi_0^2 \tag{T12.10}$$

which is the same as the average kinetic energy in Eq T12.4.

Total energy density

We find that the potential energy density in the sound wave is equal to the kinetic energy density.

Overall, then, the energy density in a sound wave is

$$\langle E_\mathrm{tot}\rangle = \frac{1}{2}\rho_0\omega^2\xi_0^2 = \frac{1}{2}Z\omega^2\xi_0^2/c. \tag{T12.11}$$

You may see expressions written in terms of rms (root mean square) values. Here

$$\xi_\mathrm{rms}^2 = \frac{1}{\lambda}\int_0^\lambda \xi(x,t)^2 dx$$

and I will leave it to you to show that

$$\xi_\mathrm{rms} = \frac{1}{\sqrt{2}}\xi_0.$$

So far, this is similar to the Simple Harmonic Oscillator. There is one significant difference, though. Remember for the Simple Harmonic Oscillator that the energy swapped between Kinetic Energy and PE, and the two were out of phase. In the wave, both Kinetic Energy and Potential Energy involve $\sin^2(\omega t - kx)$, that is they are in phase. The energy density varies along a wavelength, and it is the passage of this energy which drives the wave. Note, though, that in a standing wave the sum of the kinetic and potential energies is constant.

Energy flux

We know the energy density (energy per unit volume) in the wave, but we also know that the wave is travelling: the energy flux, energy per area per second, must surely be the energy per volume times the rate of movement of the energy.

The rate at which energy is transferred, the energy flux, is the product of energy density and wave velocity,

$$I = c\langle E\rangle = \frac{1}{2}\omega^2 Z\xi_0^2. \tag{T12.12}$$

Remember that the velocity with which energy is transported is the group velocity.

Examples: sound in air and water

How big are sound pressures and displacements? We know that the cone of a loudspeaker does not move very far, and this gives us the displacement amplitude. Let us suppose that $\xi_0 = 0.01$ mm. With a density of 1.3 kg m^{-3}, a wave speed of 330 m s^{-1} and a frequency of 1kHz this gives

$$I = \frac{1}{2}\rho\omega^2\xi_0^2 c = \frac{1}{2} \times 1.3 \times \left(2\pi^{\times}10^3\right)^2 \times \left(0.01 \times 10^{-3}\right)^2 \times 330 = 0.85\text{W m}^{-2}.$$

A sound intensity of order 1 Watt per square metre is quite loud. We can translate the amplitude into a pressure amplitude by using

$$p = -B_{\mathrm{a}}\frac{\partial \xi}{\partial x}$$

giving the pressure amplitude as

$$p_0 = B_{\mathrm{a}}k\xi_0 = B_{\mathrm{a}}\omega\xi_0/c = c\omega\rho_0\xi_0 = 330 \times 2\pi 10^3 \times 1.3 \times 10^{-5} \approx 27\text{Pa}.$$

This is less than 1/1000 of atmospheric pressure — the excess pressure in a sound wave is a very small fraction of the ambient pressure. The limit of audibility at the same frequency corresponds to about 10^{-10} atmospheres — the dynamic range of the ear is astounding.

In water, we can easily generate much bigger pressures. The bulk modulus for water is 2.2×10^9 Pa, and the density is 1000 kg m^{-3}. The sound velocity is then 1500 m s^{-1}. Note that this is not much bigger than that of sound in air - the extra resistance to compression of the water is balanced by its higher density. For the same sound intensity of 1 W m^{-2} the pressure amplitude at 1kHz would be

$$p_0 = Bk\xi_0 = Bk\sqrt{\frac{2\langle E\rangle}{\rho_0\omega^2}} = \frac{B}{c}\sqrt{\frac{2I}{\rho_0 c}} = 1.7 \times 10^3 \text{ Pa}$$

which is about 2 percent of an atmosphere. It is easy to achieve powers in water which bring p_0 up to about 1 atmosphere, which leads to the phenomenon of cavitation, the alternating creation and collapse of bubbles. The collapse of bubbles once the pressure becomes less negative is almost instantaneous, leading to local pressure pulses which are much greater than the pressure amplitude of the wave.