Topic 8 - Derivation of Wave Equation - Continuous String

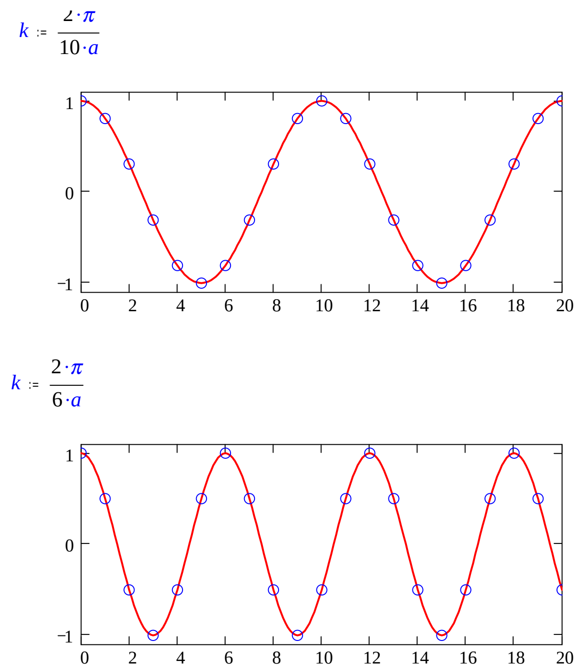

Figure T8.1: A single-frequency wave on a beaded string.

We have seen that a string under tension, regularly loaded with masses, can support a wave-like motion. For reasonably long wavelengths (figure T8.1) the wave-like pattern is quite clear. At the minimum wavelength, however, which is twice the bead spacing, alternate masses move up and down (figure T8.2) — without the continuous line joining the masses one would hardly see this as a sinusoidal wave.

T8.1 Limiting form – wave equation

We drew an analogy between waves on a beaded string and waves in a crystal. For classical wave motion, we should be able to ignore the discrete, atomic, nature of material. We replace the discrete masses m by mass distributed over the length of the string, and give the string a uniform density so that

$k := \frac{2\cdot\pi}{2\cdot a}$

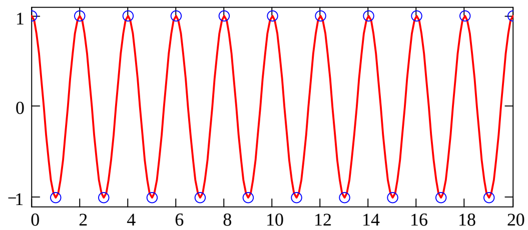

Figure T8.2: The minimum wave-length wave on a beaded string.

the mass of a length a of string will be ρa. To see what happens, go back to the equation of motion

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} = \frac{T}{ma}\left(y_{r-1} - 2y_r + y_{r+1}\right)$$

and see what happens if we allow the spacing a to become very small: call it δx. Then

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} = \frac{T}{m}\left(\frac{y_{r-1} - y_r}{\delta x} - \frac{y_r - y_{r+1}}{\delta x}\right)$$

and, as δx is small, we can replace

$$\frac{y_{r+1} - y_r}{\delta x} = \frac{y(r\delta x + \delta x) - y(r\delta x)}{\delta x} = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=(r+1/2)\delta x}$$

so that

$$\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} = \frac{T}{m}\left(\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=(r+1/2)\delta x} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=(r-1/2)\delta x}\right).$$

But, of course,

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=(r+1/2)\delta x} - \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)_{x=(r-1/2)\delta x} = \delta x\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right)_{x=r\delta x}$$

so if we smear the mass out, so that

$$m = \rho\delta x$$

we have

$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho}\frac{\partial^2 y}{\partial x^2}.$$

We have replaced the total derivatives by partial derivatives because we have replaced the discrete labels r for the position along the string by the continuous variable x.

Thinking back to the discrete case, we said that the wave speed was $\sqrt{T/(m/a)}$, so here the speed is $\sqrt{T/\rho}$.

Also note that as δx is infinitesimal, there is no cut-off wavelength, and we will always be in the regime 'wavelength large compared to details of structure', that is, in the non-dispersive regime.

T8.2 Transverse waves on a stretched string - direct approach *FGT380, AF761-763, AF919-921,*

In our derivation of the wave equation for the string, we crept up on the continuum equation by taking it as the limit of a string with masses on it. Now let's do the same job by treating it as a continuous object (see figure T8.3) of mass ρ per unit length under tension T.

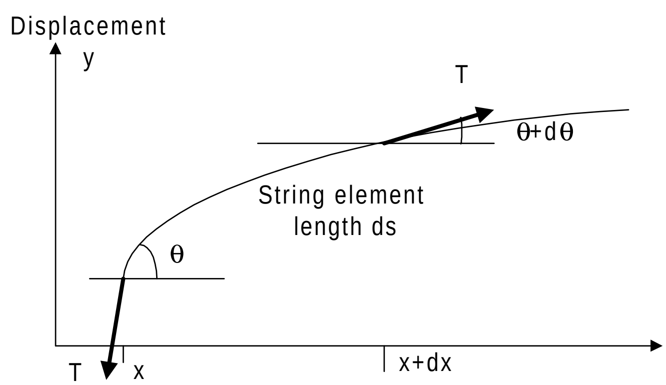

Figure T8.3: Forces on an element of a stretched string.

The length of the segment of string between x and $x + \mathrm{d}x$ is

$$\mathrm{d}s = \sqrt{\mathrm{d}x^2 + \mathrm{d}y^2} = \mathrm{d}x\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2} = \mathrm{d}x$$

provided that the slope is small.

The transverse force on the element ds is $T\sin(\theta+\mathrm{d}\theta) - T\sin(\theta)$, but as θ is small we may equate the sin to the tan, which is the gradient, so the force is

$$T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\mathrm{d}x} - \left(\frac{\partial y}{\partial x}\right)_{x}\right] = T\frac{\partial^2 y}{\partial x^2}\mathrm{d}x.$$

wave equation *FGT380, AF761-763*

We can then write the equation of motion

$$\begin{array}{ccc} \text{Force} & = & \text{Mass} \times \text{Acceleration} \\ T\dfrac{\partial^2 y}{\partial x^2}\mathrm{d}x & = \rho\mathrm{d}x & \dfrac{\partial^2 y}{\partial t^2} \\ \dfrac{\partial^2 y}{\partial x^2} & = \frac{\rho}{T} & \dfrac{\partial^2 y}{\partial t^2} \\ \dfrac{\partial^2 y}{\partial t^2} & = \frac{T}{\rho} & \dfrac{\partial^2 y}{\partial x^2}, \end{array}$$

that is, a wave equation with wave speed $c = \sqrt{T/\rho}$.

One possible source of confusion arises when we have a mechanical disturbance. Two velocities are involved - the velocity of the particles, which in the case of the wave on the string is always *perpendicular* to the string, and varies with time and position, and the velocity of the wave itself, which is parallel to the string. The velocity of the string is always transverse, and is given by $\partial y/\partial t$, i.e.

$$\begin{aligned} y(x,t) &= \mathrm{Re}\left[a e^{i(\omega t - kx)}\right] \\ &= a\cos(\omega t - kx) \quad \text{for real } a \\ \frac{\partial y}{\partial t} &= \mathrm{Re}\left[i\omega a e^{i(\omega t - kx)}\right] \\ &= -\omega a \sin(\omega t - kx). \end{aligned}$$

In other words, the transverse velocity depends on the frequency and the amplitude, and varies with time. The wave velocity is a constant: in a linear wave (the only sort we deal with) it is independent of amplitude, although (dispersion) it may depend on frequency.

Mathematics used in this topic

Fundamental definition of a derivative:

$$\frac{\mathrm{d}}{\mathrm{d}z}f(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

from which, by finding the first derivatives at two points a distance δz apart, it follows that the second derivative is

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}f(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - 2f(z) + f(z-\delta z)}{(\delta z)^2}.$$

Whether we take a 'one-sided difference'

$$\frac{f(z+\delta z) - f(z)}{\delta z} \approx \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z}$$

or a 'centred difference'

$$\frac{f(z+\delta z) - f(z)}{\delta z} \approx \left.\frac{\mathrm{d}f}{\mathrm{d}z}\right|_{z+\frac{1}{2}\delta z}$$

of course makes no difference in the limit $\delta z \to 0$.