Topic 7 - Derivation of Wave Equation - Beaded String

In a single oscillator (a pendulum, say) energy is constantly exchanged between potential and kinetic forms. The energy, though, stays in one *place* — where the oscillator is. If we couple several pendulums[1] together it is possible for energy to move from one to another, and this is essentially what wave motion is (some additional material on coupled oscillators is available at T7.4).

T7.1 Mass on a string – transverse vibrations

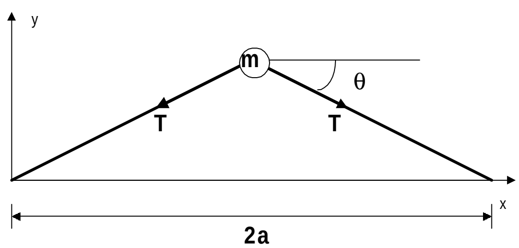

Figure T7.1: A single bead on a stretched wire acting as a simple harmonic oscillator.

To start with, look at a single mass on a string stretched under tension T between two points a distance $2a$ apart (see figure T7.1). If the sideways displacement of the mass, y, is not too great, so that the angle θ made by the string with the horizontal is small, we can write the equation of motion of the mass as

$$\begin{aligned} m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + 2T\sin\theta &= 0 \\ \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \frac{2T}{ma}y &= 0, \end{aligned}$$

[1]For pedants who object that I should use the plural 'pendula' I offer the story of an academic who invited a colleague to tea on Sunday so that they could 'discuss some conundra concerning pendula.' His colleague declined, at the same time hoping that 'we can find some better way of spending our weekend than sitting on our ba doing sa.'

where we have used the fact that for small angles

$$\sin(\theta) \approx \theta \approx \tan(\theta) = \frac{y}{a}.$$

This is just the equation for simple harmonic motion,

$$\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + \omega_0^2 y = 0,$$

thus a single mass oscillates with characteristic frequency given by

$$\omega_0^2 = 2T/ma.$$

T7.2 Regularly loaded string *P84-90*

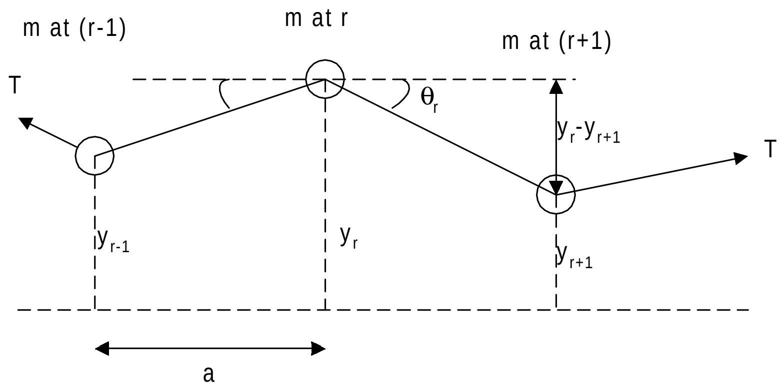

Figure T7.2: Displacements on a regularly beaded string.

Now build a regularly loaded string by putting N equal masses m at equal horizontal spacings a, as in figure T7.2. Put the string under constant tension T. Now focus in on one section of string, with masses labelled $r-1$, r and $r+1$. Then mass r is pulled towards the undisplaced position by forces $T\sin\theta_{r-1}$ and $T\sin\theta_r$, so its equation of motion is (again using $\sin(\theta) \approx \theta \approx \tan(\theta)$)

$$\begin{aligned} m\frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} &= -T(\sin\theta_{r-1} + \sin\theta_r) \\ &= -T\left(\frac{y_r - y_{r-1}}{a} + \frac{y_{r+1} - y_r}{a}\right) \\ \frac{\mathrm{d}^2 y_r}{\mathrm{d}t^2} &= \frac{1}{2}\omega_0^2\left(y_{r-1} - 2y_r + y_{r+1}\right). \end{aligned} \qquad \text{(T7.1)}$$

Let us ask whether a wave-like motion will propagate along this chain. As we are only 'sampling' the disturbance at the positions of the masses, we will replace the position x with the position of a typical mass, ra. Then we set

$$y_r = Ae^{i(kra-\omega t)}.$$

Straightforward subsitution in equation T7.1 gives

$$-\omega^2 Ae^{i(kra-\omega t)} = \frac{1}{2}\omega_0^2\left[Ae^{i(k(r-1)a-\omega t)} - 2Ae^{i(kra-\omega t)} + Ae^{i(k(r+1)a-\omega t)}\right],$$

which we may divide by $Ae^{i(kra-\omega t)}$ to obtain

$$\begin{aligned}
-\omega^2 &= \frac{1}{2}\omega_0^2\left[e^{-ika} - 2 + e^{ika}\right] \\
&= \frac{1}{2}\omega_0^2(2\cos(ka) - 2) \\
\omega^2 &= \omega_0^2(1 - \cos(ka)) \\
&= 2\omega_0^2 \sin^2\left(\frac{ka}{2}\right).
\end{aligned}$$

This shows that we *can* have a wave-like motion, and it gives us an equation relating the frequency of the wave to its wave-vector — an equation quite different from $\omega = ck$.

T7.3 Dispersion relation

Note that there is a functional dependence of ω on k which leads to a velocity ω/k which depends on the frequency. This is the phenomenon of *dispersion* which we have already discussed. In fact waves in a crystal have dispersion relations which are very similar to the one we have just described. Furthermore, the fact that

$$\cos(ka) = 1 - \frac{\omega^2}{\omega_0^2}$$

tells us that ω/ω_0 is limited between 0 and $\sqrt{2}$. There is an upper cutoff frequency above which a wave cannot be transmitted through the system. Of course, when ω takes this maximum value, $\cos(ka) = -1$ which corresponds to successive masses moving in antiphase. We clearly can't get a wave with any shorter wavelength than this, because we don't have beads to 'mark' the displacement at any shorter spacing. At this minimum wavelength, $ka = \pi$, or $\lambda = 2a$.

For long wavelengths, though (that is, small ka) we may expand

$$\sin\left(\frac{ka}{2}\right) \approx \frac{ka}{2}$$

so that

$$ka = \sqrt{2}\omega/\omega_0$$

or the wave speed

$$\frac{\omega}{k} = \frac{a\omega_0}{\sqrt{2}} = \sqrt{\frac{2T}{ma}}\frac{a}{\sqrt{2}} = \sqrt{\frac{T}{m/a}}.$$

In this limit, then, we have regained a non-dispersive wave, with the frequency directly proportional to the wavevector.

T7.4 Supplementary Material - Coupled Pendulums

two stiff pendulums, joined by spring

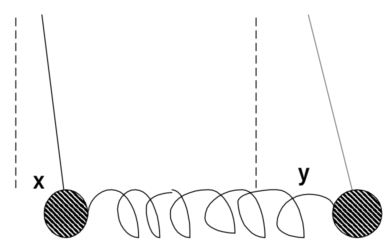

Figure T7.3: Two pendulums, coupled by a spring.

Consider two identical pendulums (figure T7.3), each with a mass m on the end of a light rod of length l. Hang them side by side, and join the masses by a light spring, which is unstretched when the pendulums hang vertically. Let the displacements of the masses be x and y, so that the spring will be stretched by $(x - y)$. In the usual small oscillation approximation we have

$$\begin{aligned} m\frac{\mathrm{d}^2x}{\mathrm{d}t^2} &= -mg\frac{x}{l} - k(x-y) \\ m\frac{\mathrm{d}^2y}{\mathrm{d}t^2} &= -mg\frac{y}{l} + k(x-y) \end{aligned}$$

or, substituting in for the fundamental frequency of the pendulums

$$\begin{aligned}\frac{d^2x}{dt^2} &= -\omega_0^2 x - \frac{k}{m}(x-y)\\ \frac{d^2y}{dt^2} &= -\omega_0^2 y + \frac{k}{m}(x-y).\end{aligned}$$

normal coordinates

The equations are more neatly solved if we take the sum and difference of these equations:

$$\begin{aligned}\frac{d^2}{dt^2}(x+y) &= -\omega_0^2(x+y)\\ \frac{d^2}{dt^2}(x-y) &= -\left(\omega_0^2 + 2\frac{k}{m}\right)(x-y).\end{aligned}$$

What this shows us is that whereas the motions of x and y are coupled, if we define *normal coordinates* by

$$\begin{aligned}X &= x+y\\ Y &= x-y\end{aligned}$$

and the motions of these coordinates are not coupled:

$$\begin{aligned}\frac{d^2X}{dt^2} &= -\omega_0^2 X\\ \frac{d^2Y}{dt^2} &= -\left(\omega_0^2 + 2\frac{k}{m}\right)Y.\end{aligned}$$

If $Y = 0$, the pendulums swing together and the spring does not alter in length; as a result, of course, the frequency is unaltered. On the other hand, if $X = 0$, the spring stretches and extends as the pendulums swing, and the frequency is increased to ω' where $\omega'^2 = \omega_0^2 + 2k/m$.

energy interchange

Of course, any motion of the system may be represented by

$$\begin{aligned}X = x+y &= X_0\cos(\omega_0 t + \phi_1)\\ Y = x-y &= Y_0\cos(\omega' t + \phi_2)\end{aligned}$$

as an example, take both X_0 and Y_0 as a, both phases to be 0, then

$$x = \frac{1}{2}(X+Y) \quad = \quad \frac{1}{2}a(\cos(\omega_0 t) + \cos(\omega' t))$$
$$y = \frac{1}{2}(X-Y) \quad = \quad \frac{1}{2}a(\cos(\omega_0 t) - \cos(\omega' t))$$

but these are expressions we have seen before, giving beats:

$$x \quad = \quad 2\cos\frac{(\omega' - \omega_0)t}{2}\cos\frac{(\omega' + \omega_0)t}{2}$$
$$y \quad = \quad 2\sin\frac{(\omega' - \omega_0)t}{2}\sin\frac{(\omega' + \omega_0)t}{2}.$$

In fact, if we look at the displacements in more detail (as in figure T7.4, we see that both the carriers and the envelope functions for the two pendulums are exactly out of phase. The energy is regularly interchanged between the two. Of course, once the energy has moved from one pendulum to the other the only place it can go is back again (effectively, if we think of it as a wave, it is reflected backwards and forwards). If we add more pendulums, though, it can carry on down the line - which will give a wave.

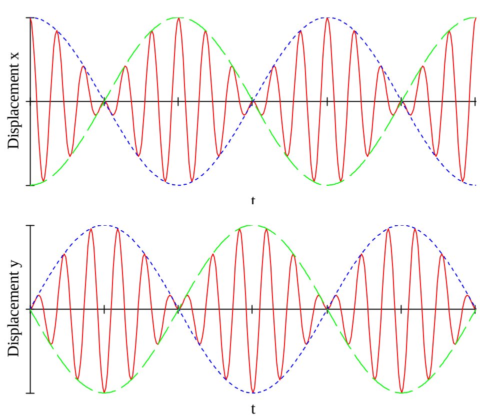

Figure T7.4: Interchange of energy between two coupled pendulums.

We have seen how it is possible for linked oscillators to transfer energy amongst themselves. We now move on to look at how this energy transfer can propagate through a series of oscillators to form a wave.