Topic 4 - The Wave Equation – Basic Properties

In this lecture we introduce the wave equation — the universal form of equation which governs all the wave behaviour we are going to cover in this course. We will go on to talk about general forms of solution, again derving results which will be relevant to all the waves we meet later, be they mechanical waves, light waves, or any other waves.

T4.1 The one-dimensional wave equation *FGT380-386, AF749-754, AF919- 921*

The wave equation is a *linear second order partial differential equation.* For a disturbance ψ, which might represent the pressure in a sound wave, or the electric field in a radio wave, the amount of disturbance varies from place to place and from time to time. That is, ψ is a function of a spatial variable, x, and time, t. The wave equation is

$$\frac{\partial^2\psi(x,t)}{\partial t^2} = c^2\frac{\partial^2\psi(x,t)}{\partial x^2}.$$

Let us explain the words in the description:

- *linear* because ψ appears to the first power (linearly) throughout the equation;
- *second order* because the highest (indeed, the only) derivatives which appear are ∂^2;
- *partial* because ψ is a function of two variables, x and t, and therefore we have to use partial rather than total derivatives.

The constant c is the speed of the wave.

$h(x-ct)+g(x+ct)$

The most general form of solution has the form

$$\psi(x,t) = h(x-ct)+g(x+ct),$$

where h and g are *any* continuous, twice-differentiable functions.

Using the shorthand

$$\frac{\mathrm{d}}{\mathrm{d}z}h(z) = h'(z)$$

and the corresponding double-prime notation for the second derivative, we can see by substitution that this solution is valid(see Maths used in this Topic, section T4.2, for a fuller explanation):

$$\begin{aligned}
\frac{\partial \psi}{\partial x} &= h'(x-ct) + g'(x+ct) \\
\frac{\partial \psi}{\partial t} &= -ch'(x-ct) + cg'(x+ct) \\
\frac{\partial^2 \psi}{\partial x^2} &= h''(x-ct) + g''(x+ct) \\
\frac{\partial^2 \psi}{\partial t^2} &= c^2h''(x-ct) + c^2g''(x+ct).
\end{aligned}$$

What do these solutions represent? As shown in figure T4.1, $h(x-ct)$ represents a disturbance, with a shape given by h, which moves along the x axis in the positive direction (because the profile at time t is exactly the same as the profile at time 0, but moved a distance ct along the x axis. Similarly, $g(x+ct)$ represents a pulse propagating along the negative axis.

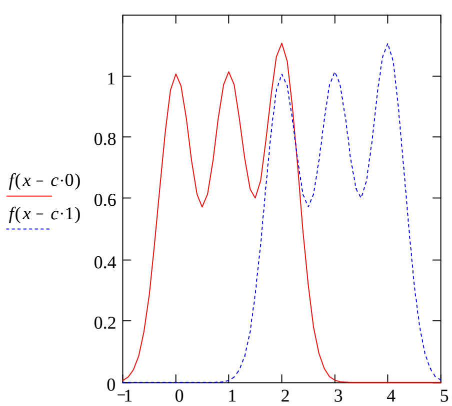

Figure T4.1: The propagation of a general wave pulse.

As an example, suppose our function h is a so-called Lorentzian function,

$$h(x-ct) = \frac{B}{b+(x-ct)^2}$$

(clearly if this represented transverse waves on a string, B would have dimensions of $length^3$, b would have dimensions of $length^2$, so that the overall h would be a distance, the distance by which the string is pulled aside. Going through the maths,

$$\begin{aligned}
\frac{\partial}{\partial x}\frac{B}{b+(x-ct)^2} &= \frac{-2B(x-ct)}{\left[b+(x-ct)^2\right]^2} \\
\frac{\partial^2}{\partial x^2}\frac{B}{b+(x-ct)^2} &= \frac{-2B}{\left[b+(x-ct)^2\right]^2} + \frac{2\times 2\times 2B(x-ct)^2}{\left[b+(x-ct)^2\right]^3} \\
\frac{\partial}{\partial t}\frac{B}{b+(x-ct)^2} &= \frac{2cB(x-ct)}{\left[b+(x-ct)^2\right]^2} \\
\frac{\partial^2}{\partial t^2}\frac{B}{b+(x-ct)^2} &= \frac{-2Bc^2}{\left[b+(x-ct)^2\right]^2} + \frac{2\times 2\times 2Bc^2(x-ct)^2}{\left[b+(x-ct)^2\right]^3}.
\end{aligned}$$

and clearly

$$\frac{\partial^2}{\partial t^2}\frac{B}{b+(x-ct)^2} = c^2\frac{\partial^2}{\partial x^2}\frac{B}{b+(x-ct)^2}.$$

The crucial point, though, is that *any* function of $x-ct$ or $x+ct$ will satisfy the wave equation.

As we said before, the symbol ψ may represent all sorts of different quantities. In a wave on a string, it could be the displacement of the string sideways; for a wave on the surface of water it would be the vertical movement of a small volume of water; for sound waves in a gas it could be the change in pressure; for light or radio waves it could be the electric field. Note that as ψ appears in a similar position on each side of the equation, the equation does not depend on the *dimensions* of ψ.

Use of single-frequency waves

Although the general solutions show the nature of waves, and the propagation of pulses, very neatly, we more frequently work with waves of specified frequency. Why?

- Our input (musical note, light from a radiating atom, radio signal,...) is often close to a single frequency;
- Material properties often vary with frequency, so it's useful to work with signals tagged by frequency;
- We can synthesise a general signal from a collection of signals of different frequencies (Fourier series and transforms - which you have seen demonstrated in the computing course and will again meet next year). Figure T4.2 shows the way a series of sinusoidal functions can be combined to make an different shape, in this case a square wave. For this to work, though, the *linearity* of the wave equation is essential.

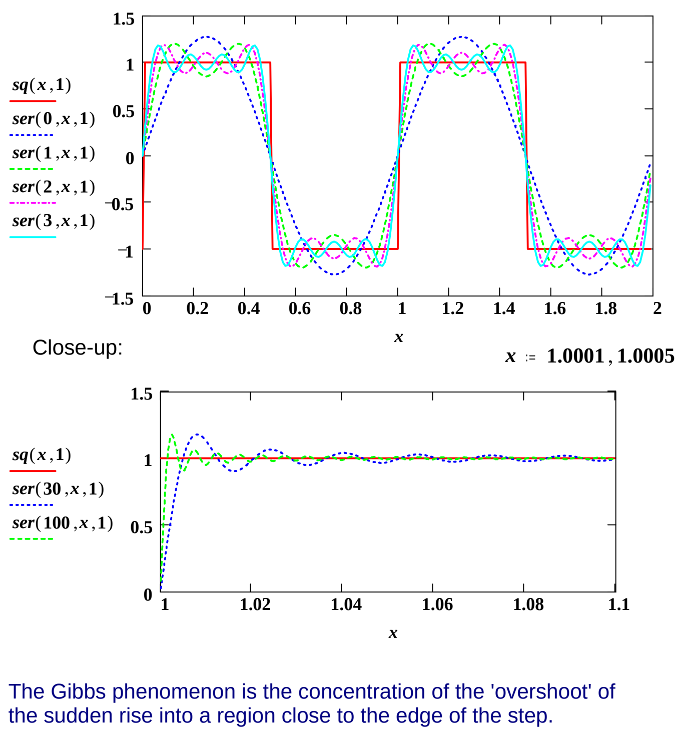

Figure T4.2: Demonstration of Fourier's theorem: sine waves are being superposed to make a square wave.

T4.2 Mathematics used in this topic

The trickiest point to grasp is the differentiation of the general solution of the wave equation. Taking a rightward-travelling wave $h(x - ct)$, think of this as a function h of another function $u(x, t) = x - ct$. Then to differentiate h we can use the chain rule

$$\frac{\partial}{\partial t} h(u(x, t)) = \frac{\partial u}{\partial t} \frac{\mathrm{d}h}{\mathrm{d}u}.$$

Note that as h is being written as a function of just *one* variable, u, the derivative of h with respect to u is a *total* derivative, denoted by d instead of ∂. On the other hand, u is a function of the *two* variables x and t, and we have to use partial derivatives for the derivatives of u.

The partial derivative of $u(x, t) = x - ct$ with respect to t is easy — it just gives $-c$. Then, if we use a shorthand notation for the derivative of h

$$\frac{\mathrm{d}h}{\mathrm{d}u} = h'$$

we have

$$\frac{\partial}{\partial t} h(u(x, t)) = -c \times h'(u(x, t)) = -ch'(x - ct).$$

It should be clear what the corresponding result is for the x derivative, and how to carry this on to the second derivative:

$$\frac{\partial}{\partial x} h(u(x, t)) = \frac{\partial u}{\partial x} \frac{\mathrm{d}h}{\mathrm{d}u} = 1 \times h'(u(x, t)) = h'(x - ct).$$

Now both the first derivatives are functions of u only, so the pattern repeats:

$$\frac{\partial^2}{\partial t^2} h(u(x, t)) = \frac{\partial}{\partial t} \frac{\partial h}{\partial t} = \frac{\partial}{\partial t}(-c)h'(u) = -c \frac{\partial u}{\partial t} \frac{\mathrm{d}h'}{\mathrm{d}u} = c^2 h'',$$

and similarly for the second derivative with respect to x.