Topic 2 – Complex Numbers; Energy of Oscillators

T2.1 Simple Harmonic Oscillations

The last lecture showed the relationship between simple harmonic motion and motion in a circle - that is, motion in a plane.

relation to complex numbers *FGT916*

We know that a compact notation for describing motion in two dimensions (a plane) exists, in the form of complex numbers. We can relate this to the problem of motion in a circle simply through de Moivre's theorem:

$$e^{i\theta} = \cos\theta + i\sin\theta,$$

so that we may write our coordinates in the plane in the form $x + iy$:

$$\begin{aligned} x + iy &= a\cos(\omega_0 t + \phi) + ia\sin(\omega_0 t + \phi) \\ &= ae^{i(\omega_0 t + \phi)}. \end{aligned}$$

N.B. Sign convention - some people write $e^{-i\omega_0 t}$, and engineers use $e^{\pm j\omega_0 t}$ - but the sign does not matter because all that matters is the projection of this onto one of the axes. If we take the x axis as before, we have

$$x(t) = a\cos(\omega_0 t + \phi) = Re\left[ae^{i(\omega_0 t + \phi)}\right]$$

as before. Note that a was real in the cos/sin formulation, and is real in the complex exponential form too. It is simply the amplitude of the oscillation. If we like we can combine the amplitude and phase information into one constant, by defining

$$A = ae^{i\phi}.$$

Why bother with complex notation, then? Simply because complex exponentials are so easy to manipulate. I know, for example, that

$$\left[e^{i\omega_0 t}\right]^4 = e^{4i\omega_0 t},$$

but I would have trouble in working out what

$$[\cos(\omega_0 t)]^4$$

would be in terms of multiple angles – actually it's

$$\frac{1}{8}\left((3 + 4\cos(2\omega_0 t) + \cos(4\omega_0 t)\right).$$

What is more, when we come to differentiate expressions describing the behaviour of oscillators, simplifications occur. For example, the velocity of an oscillator (the derivative with respect to time of its displacement) contains exactly the same complex exponential as the displacement:

$$\begin{aligned} y(t) &= ae^{i\omega t} \\ \dot{y}(t) &= i\omega a e^{i\omega t} \end{aligned}$$

and so the time-varying term $e^{i\omega t}$ can be factored out of expressions which link displacement and velocity. If we use trigonometric forms

$$\begin{aligned} Y(t) &= a\cos(\omega t) \\ \dot{Y}(t) &= -\omega a \sin(\omega t) \end{aligned}$$

this is no longer the case.

I'll try to show, at various points in the course, where the maths is being made easier by using the complex exponential notation.

Any physical quantity, however, is real, and is obtained from the complex form by taking the Real part

T2.2 Interchange of energy *AF194-196*

We saw in the previous topic that if we take an oscillator then energy is exchanged between kinetic and potential forms. We'll look at this again in the complex number representation.

Consider a pendulum, length l, with a bob of mass m and a massless rod. The equation of motion is, resolving forces and accelerations in the direction perpendicular to the string,

$$-mg\sin\theta = ma_{\mathrm{T}}.$$

Now $a_{\mathrm{T}} = l\mathrm{d}^2\theta/\mathrm{d}t^2$, and if the angle of swing is small the angle of the rod to the vertical, θ, obeys

$$ml\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + mg\theta = 0$$

or

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} + \omega_0^2\theta = 0$$

where

$$\omega_0^2 = \frac{g}{l}.$$

We may therefore write

$$\theta(t) = \mathrm{Re}\left[\theta_0 e^{i(\omega_0 t+\phi)}\right].$$

Remember that θ_0 is real, so

$$\theta(t) = \theta_0 \mathrm{Re}\left[e^{i(\omega_0 t+\phi)}\right].$$

When we get more used to using this notation, we may drop the Re[], but for the moment we keep it explicit.

What is the angular velocity?

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \theta_0 \mathrm{Re}\left[i\omega_0 e^{i(\omega_0 t+\phi)}\right].$$

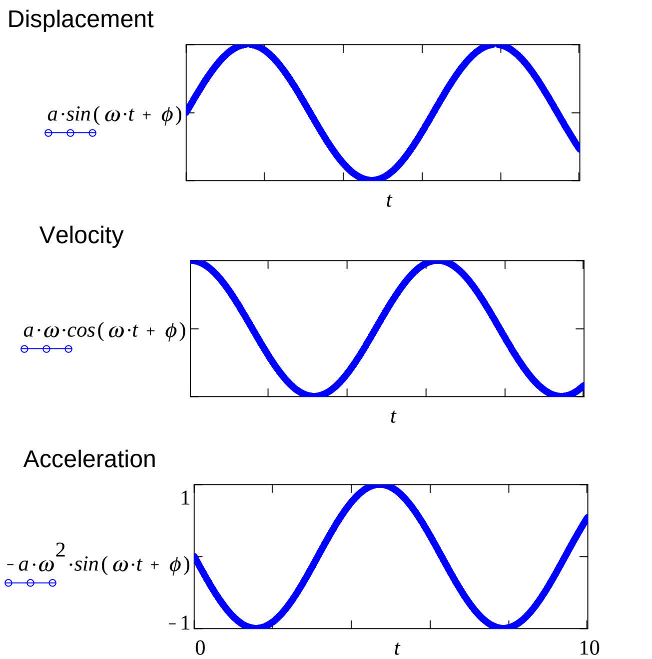
ω_0 is real, so

$$\begin{aligned}\frac{\mathrm{d}\theta}{\mathrm{d}t} &= \theta_0\omega_0 \mathrm{Re}\left[i e^{i(\omega_0 t+\phi)}\right] \\ &= -\theta_0\omega_0 \sin(\omega_0 t+\phi).\end{aligned}$$

This is just what we would have got by using cosine form, of course. Note, when we look back at the complex form, that the change from cos to sin, which in terms of the phasor is represented by a rotation through 90 degrees, is represented by i - which is as we expect, as multiplying a number by i rotates it by 90 degrees in the complex plane. We say that the velocity is *in quadrature* with the position: see Figure T2.1.

energy - need to use real parts of complex terms

Now let us ask about the energy of this system. Consider the kinetic energy first. The velocity of the pendulum bob is $l\frac{\mathrm{d}\theta}{\mathrm{d}t}$ so the kinetic energy is



Displacement lags behind velocity by π/2 radians, an
lags behind acceleration by π radians.

The phase constant φ has been taken to be zero.

Figure T2.1: Relationship between a rotating vector and simple harmonic motion.

$\frac{1}{2}ml^2\left(\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2$. Of course the energy must be real, and to compute it we have to extract the real velocity: that is

$$\begin{aligned} T(t) &= \frac{1}{2}ml^2\theta_0^2\left[\mathrm{Re}\left[i\omega_0 e^{i(\omega_0 t+\phi)}\right]\right]^2 \\ &= \frac{1}{2}ml^2\omega_0^2\theta_0^2\left[\sin(\omega_0 t+\phi)\right]^2 \\ &\neq \frac{1}{2}ml^2\omega_0^2\theta_0^2\mathrm{Re}\left[\left[ie^{i(\omega_0 t+\phi)}\right]^2\right] \\ &\neq \frac{1}{2}ml^2\omega_0^2\theta_0^2\left|ie^{i(\omega_0 t+\phi)}\right|^2 \end{aligned}$$

Similarly, the potential energy of the mass is

$$V(t) = -mgl\left(\cos(\theta)-1\right)$$

$$\begin{aligned} &\approx \frac{1}{2} m g l \theta^2 \\ &= \frac{1}{2} m g l \theta_0^2 \left[\cos(\omega_0 t + \phi)\right]^2 \end{aligned}$$

and the sum $T(t) + V(t)$, using $g/l = \omega_0^2$, is

$$E(t) = mgl\theta_0^2 = ml^2\omega_0^2\theta_0^2,$$

again, a constant.

Phasors and Complex Numbers

At first sight it appears that by taking only the real part of a complex number to represent the physical disturbance, we must be throwing something away. This is not the case – and there are two ways of seeing this. The first is to remember how we got into complex numbers in the first place. We came in through phasors, when it was the projection of the rotating vector onto one Cartesian axis that represented the simple harmonic motion, and the projection onto the other axis was not used. In just the same way, it is the projection onto one axis in the complex plane (conventionally the real axis) which represents the simple harmonic motion, and the projection on the other axis is not used. Looking at the situation from another point of view, whatever the projection on the real axis does, the projection on the imaginary axis does a quarter of a period later – there is no additional information in the projection on the imaginary axis.

Root Mean Square values

The properties of oscillators and waves are sometimes expressed in terms of root mean square (rms) quantities. The root mean square value of a quantity, say a displacement, is obtained by taking the square root of the average of the square: as the oscillator repeats periodically, we may take the average over one period:

$$\begin{aligned} \theta_{\rm rms}^2 &= \frac{\int_0^\tau \theta(t)^2 {\rm d}t}{\tau} \\ &= \frac{\int_0^\tau \theta_0^2 \cos(\omega t + \phi)^2 {\rm d}t}{\tau} \\ &= \frac{\int_0^\tau \theta_0^2 \frac{1}{2}\left[\cos(2(\omega t + \phi)) + 1\right] {\rm d}t}{\tau} \end{aligned}$$

$$= \frac{1}{2}\theta_0^2$$

that is

$$\theta_{\rm rms} = \frac{\theta_0}{\sqrt{2}}.$$

T2.3 Mathematics used in this Topic

$$\frac{\mathrm{d}}{\mathrm{d}x}e^{ax} = ae^{ax}$$

$$\cos^2(x) = \frac{1}{2}(\cos(2x) + 1)$$

$$\int_0^\tau \cos(n\omega t)\mathrm{d}t = 0$$

if

$$\omega = \frac{2\pi}{\tau}$$

.

De Moivre's theorem, and formulae deduced from it.

$$\cos(x) + i\sin(x) = e^{ix}$$

$\mathrm{Re}(e^{ix}) = \cos(x)$ and $\mathrm{Im}(e^{ix}) = \sin(x)$
$\mathrm{Re}(e^{-ix}) = \cos(x)$ and $\mathrm{Im}(e^{-ix}) = -\sin(x)$

T2.4 Supplementary Material: not covered in lectures

LC circuit - magnetic and electrostatic energy

We can carry out a similar analysis of the resonant LC circuit. You will be familiar with a capacitor, which has the property that $V = q/C$. Less familiar is the inductance, for which $V = L\mathrm{d}I/\mathrm{d}t$. In a closed circuit, then

$$L\frac{\mathrm{d}I}{\mathrm{d}t} + \frac{q}{C} = 0$$

but charge is conserved, so

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\mathrm{d}^2q}{\mathrm{d}t^2}$$

and we have resonance, with

$$\omega_0^2 = \frac{1}{LC}.$$

If the charge on the capacitor is

$$q(t) = q_0 \mathrm{Re}\left[ie^{i(\omega_0 t + \phi)}\right]$$

then the electrostatic energy stored in the capacitor is

$$E_{\mathrm{es}}(t) = \frac{1}{2}\frac{q_0^2}{C}\left[\cos(\omega_0 t + \phi)\right]^2$$

and the rest of the field is stored as magnetic energy (see 1B26 course)

$$E_{\mathrm{ms}} = \frac{1}{2}LI^2$$

or

$$E_{\mathrm{ms}}(t) = \frac{1}{2}\omega_0^2 q_0^2 L\left[\sin(\omega_0 t + \phi)\right]^2$$

which gives a total energy

$$E(t) = \frac{q_0^2}{2C} = \frac{1}{2}\omega_0^2 q_0^2 L,$$

or, equivalently,

$$E(t) = \frac{I_0^2}{2\omega_0^2 C} = \frac{1}{2}I_0^2 L.$$