Topic 26 — Diffraction (concluded)

finite size of slit

For the diffraction grating, then, the size of the slit applies an overall multiplicative factor to the pattern from the slits. Note that if a zero of the slit function coincides with a peak of the grating pattern we will lose one peak (actually, as a close look at the figures will show, the peak has its height substantially reduced and although its magnitude is zero at the zero of the slit function vestiges of it remain at each side) – this is known as a *missing order* in the diffraction pattern.

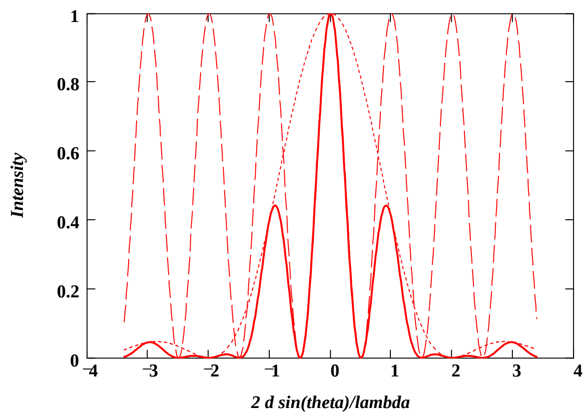

Figure L26.1: Young's slits pattern as modified by a finite slit width. In this figure the slit widths are half of the separation between the slits. Note that the even orders of diffraction are missing.

As the principal maxima occur when

$$\sin(\theta) = p\frac{\lambda}{h}$$

and the slit pattern's minima occur when

$$\sin(\theta) = p'\frac{\lambda}{d},$$

we can see that these can coincide when

$$pd = p'h,$$

or when d is a rational fraction of h. In figure L26.1 the pattern is shown for Young's slits (i.e. two slits) with the width of the slits equal to half their spacing, and all the even order principal maxima $p = 2, 4, 6, \ldots$ are missing. Remember that in a real grating the total number of slits will be very large, so the principal maxima will be very narrow, and thus the orders will only be missing if the ratio of p to p' accurately matches the ratio of h to d.

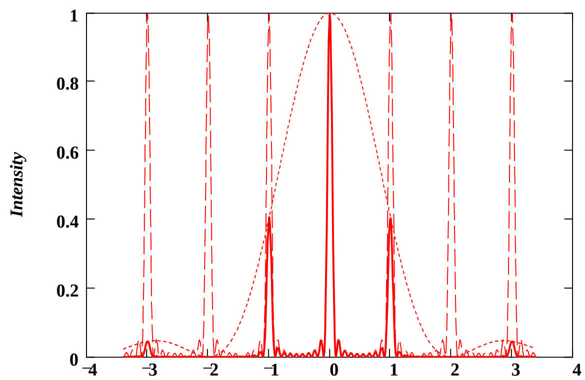

Figure L26.2: The modification of the diffraction pattern of a grating, shown for an unrepresentatively small number of slits (ten), each half as wide as the slit spacing.

Figures L26.2 and L26.3 show the effect of finite slit width on a multiple-slit diffraction grating (although here the grating is taken to have only ten slits). Note that for the narrower slits the slit diffraction pattern is wide (universal rule — narrower aperture has wider diffraction pattern), and so has less effect on the overall pattern.

L26.1 Limiting form of grating

Note that if we take the grating intensity pattern, we can recover from it the pattern of a slit. Consider what happens if we keep the slits in the very narrow, but increase their number and at the same time bring them closer together. That is, consider what happens to

$$I(\theta) = I(0) \left[\frac{\sin\left(\frac{Nkh\sin(\theta)}{2}\right)}{N\sin\left(\frac{kh\sin(\theta)}{2}\right)} \right]^2$$

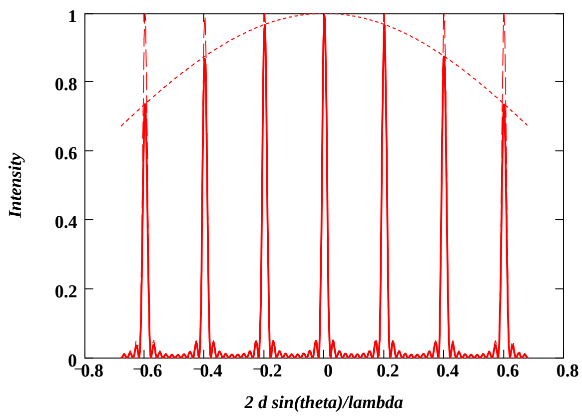

Figure L26.3: The modification of the diffraction pattern of a grating, shown for an unrepresentatively small number of slits (ten), each one tenth as wide as the slit spacing.

as we let $h \to 0$ and $N \to \inf$, but at the same time keep the overall width of the grating finite, $Nh \to d$. The sine function on the bottom line may be replaced by its argument, which is small, but the argument of the sine function on the top line is not small. Thus

$$\left[\frac{\sin\left(\frac{Nkh\sin(\theta)}{2}\right)}{N\sin\left(\frac{kh\sin(\theta)}{2}\right)}\right]^2 \to \left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{Nkh\sin(\theta)}{2}}\right]^2$$

$$\to \left[\frac{\sin\left(\frac{kd\sin(\theta)}{2}\right)}{\frac{kd\sin(\theta)}{2}}\right]^2,$$

which is exactly the diffraction pattern of a slit of width d. This should come as no surprise, as the integral as the limit of a sum is a familiar concept.

A mirror is a grating!

Within the spirit of Huygens's and Fresnel's principle, we can think of a mirror as a diffraction grating in which there are no gaps between the 'slits': every point on the mirror acts as a source of secondary wavelets. In this limit, the line spacing h is the same as the line width d. As a result, every order except the zeroth is missing - a mirror gives only one image!

L26.2 Fresnel and Fraunhofer Diffraction

We should also think about what happens at shorter ranges, where wave-front curvature is important - this is known as the Fresnel regime, whereas the region we have treated up to now, assuming plane wave fronts, is the Fraunhofer[1] regime. It is obvious that if we put a screen right up against the aperture, there is no space for a diffraction pattern to develop, and our image must be simply as wide as the slit. As we move further out, the pattern goes through a complicated intermediate regime, until we finally reach the Fraunhofer regime. Figure L26.4 illustrates the limiting cases schematically.

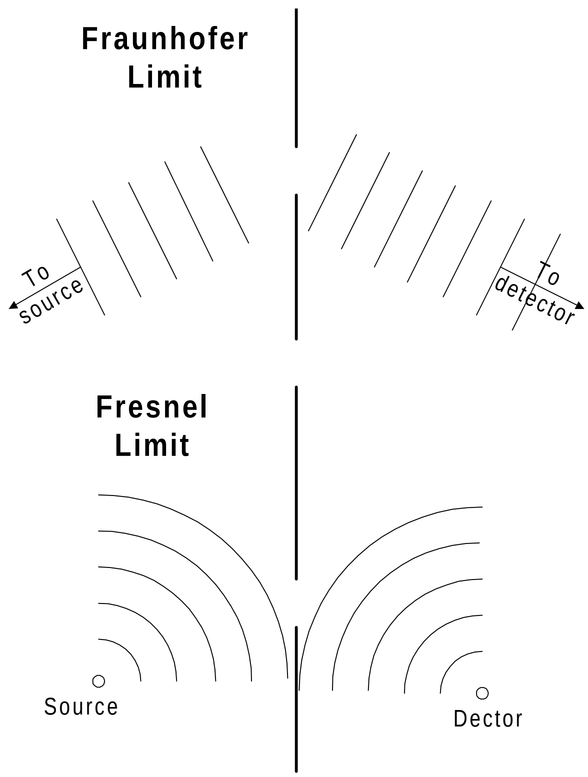

Figure L26.4: The difference between Fraunhofer and Fresnel diffraction.

Consider the path length difference between a ray from the centre and

[1] Joseph Fraunhofer (1787-1826) was a poor boy who, after several unsuccessful apprenticeships, finally came to work in the glassworks at the abbey of Benediktbeuern. The abbey's other claim to fame is that its library included the collection of poems about love, religion and student life that later became *Carmina Burana*.

the bottom of a slit of width d, observed at a position y of the axis at a distance D away along the x axis. This path difference Δ is given by

$$
\begin{aligned}
\Delta &= \sqrt{D^2 + \left(y + \frac{d}{2}\right)^2} - \sqrt{D^2 + y^2} \\
&= D\left(1 + \left(\frac{y}{D} + \frac{d}{2D}\right)^2\right)^{1/2} - D\left(1 + \frac{y^2}{D^2}\right)^{1/2} \\
&\approx D\left(1 + \frac{1}{2}\frac{y^2}{D^2} + \frac{1}{2}\frac{yd}{D^2} + \frac{1}{8}\frac{d^2}{D^2} ... - 1 - \frac{1}{2}\frac{y^2}{D^2} - ....\right) \\
&= \frac{yd}{2D} + \frac{d^2}{8D}.
\end{aligned}
$$

Now the second term is the one we have neglected so far, and arises from curvature of the wave-front. If this term is less than one eighth of a wavelength, the associated difference in phase may be neglected. Thus

the Rayleigh distance $D = d^2/\lambda$ where D is the range, d is the slit width, and λ is the wavelength, defines the distance beyond which we may treat the diffraction in the Fraunhofer approximation.

For example, with a wavelength of 500 nm and a slit width d of 0.1 mm the Rayleigh distance is 20 mm.

Fresnel regime (only qualitative results required)

In the Fresnel regime, the source and the detector are both at finite distances from the diffracting aperture. If these distances are C and D respectively, then the path length through a point $(x', y', 0)$ in the aperture will be

$$\sqrt{C^2 + x'^2 + y'^2} + \sqrt{D^2 + x'^2 + y'^2} \approx C + D + \left(x'^2 + y'^2\right)\left(\frac{1}{2C} + \frac{1}{2D}\right)$$

and so the total amplitude at the observation point will be

$$E \propto \int\int e^{-ik\left(x'^2+y'^2\right)\left(\frac{1}{2C}+\frac{1}{2D}\right)} \mathrm{d}x'\mathrm{d}y'.$$

Note again that this can be written as a product of integrals over x' and y'.

$$E \propto \int e^{-ikx'^2\left(\frac{1}{2C}+\frac{1}{2D}\right)} \mathrm{d}x' \int e^{-iky'^2\left(\frac{1}{2C}+\frac{1}{2D}\right)} \mathrm{d}y'.$$

If we write

$$s = x'\sqrt{\frac{k}{\pi}\left(\frac{1}{C}+\frac{1}{D}\right)}$$

we can write

$$E \propto \int_{s_1}^{s_2} e^{-i\pi s^2/2}\mathrm{d}s \times \text{similar integral for y direction.}$$

Separating the complex exponential into cos and sine terms, we can express the Fresnel diffraction in terms of standard integrals, known as Fresnel's integrals,

$$C(u) = \int_0^u \cos(\pi s^2/2)\mathrm{d}s$$

and

$$S(u) = \int_0^u \sin(\pi s^2/2)\mathrm{d}s$$

.

Figure L26.5 shows the evolution of the diffraction pattern of a slit as the distance form the slit increases. The key feature of the behaviour in the Fresnel zone is that a rapidly-decreasing amplitude occurs in the region of the geometric shadow, whilst in the geometrically illuminated region there are oscillations in amplitude which as largest near the edge of the shadow.

L26.3 Diffraction in two and three dimensions

We have seen from the form of the Fraunhofer diffraction expression that in a Cartesian system the diffracted amplitude may be factored into separate expressions for the different directions.

Bragg reflection *AF948*

A very important application of diffraction occurs when we have a three-dimensional array of sources - that is, a crystal. In order to see the effect we need to choose a wavelength which is comparable with the spacing between the atoms (less than half a nanometer), which is achieved with X-rays or neutrons with energies of about kT, so-called thermal neutrons.

The crystal may be thought of as a series of reflecting planes, and if the radiation is reflected at an angle θ to the planes (N.B. crystallographers use a different convention for angles) then for an interplane spacing d reflections from successive planes will be in plane if the path length difference $2d\sin(\theta)$ is a multiple of the wavelength λ.

$$\boxed{\text{Bragg's law: } 2d\sin(\theta) = n\lambda.}$$

L26.4 Resolution of images

Diffraction limits our ability to discern detail in images which we look at through an aperture - and we always do view things through an aperture, even if it is only the pupile of our eye.

The eye

How does this affect our vision? Typically the human eye has a pupil diameter (in normal light conditions) of about 2mm. If we think of yellow light, $\lambda = 550$ nm, the limit of resolution is

$$\Delta\theta = \frac{1.22 \times 550 \times 10^{-9}}{2 \times 10^{-3}} = 3.3 \times 10^{-4}\text{radians} = 0.02\text{degrees}$$

or about 1 minute of arc.

That corresponds to a separation of about 20 mm at a range of 100 metres.

This angle corresponds to a separation on the retina, about 20 mm from the lens of the eye, of about 6.7 microns. This is roughly twice the separation of the receptors on the retina - a well optimised system.

For *any* optical system, the resolution is limited by the apertures within it and the wavelength at which it operates, and may be determined by a formula similar to Rayleigh's. Telescope objectives are made large in an attempt to improve angular resolution.

Pinhole camera

Diffraction limits the behaviour of simple devices such as the pinhole camera. If we make the pinhole very small, the angular spread of the image increases ($1.22\lambda/d$). If the pinhole is larger, then the light from a point on the object cannot give a spot on the screen which is smaller than the pinhole. The best size is a compromise.

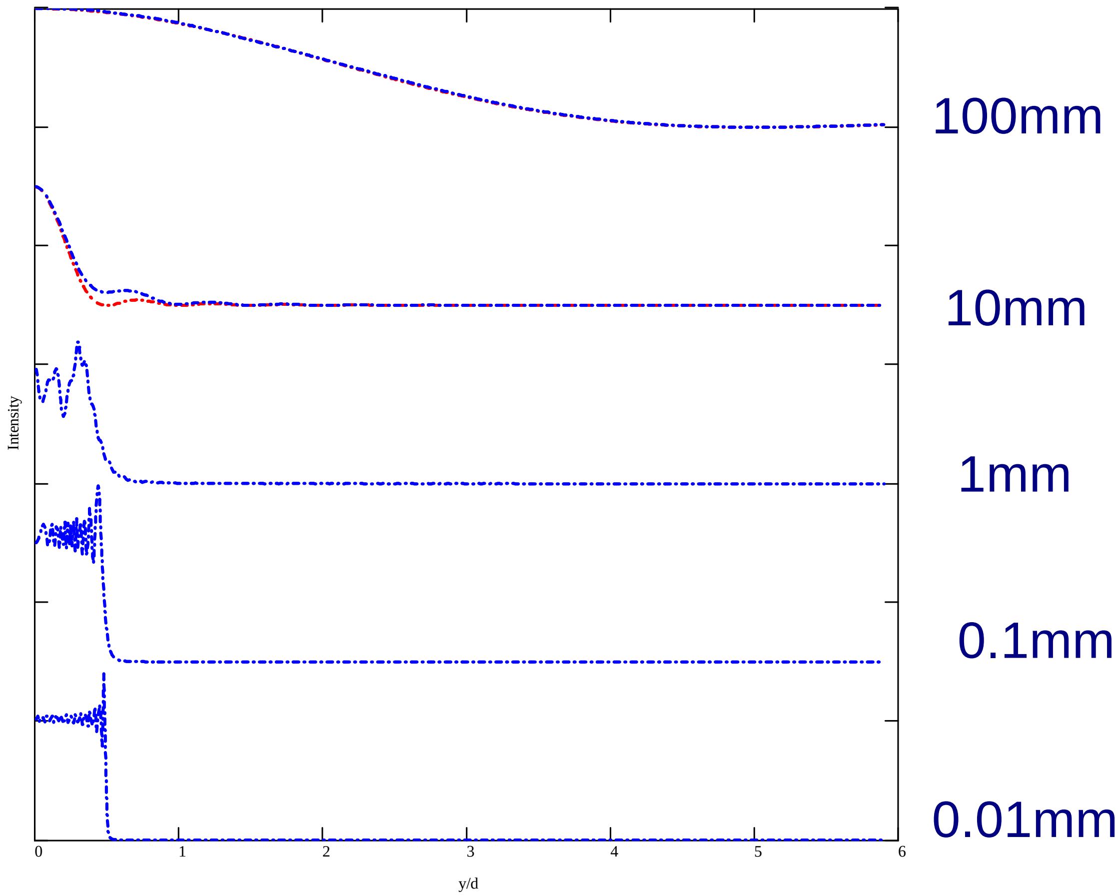

Figure L26.5: The variation of the diffraction pattern of a 0.1mm slit with 500nm light as the viewing position moves away from the slit. The Rayleigh distance is 20mm, and by that distance the Fraunhofer pattern is a good approximation to the exact one.