Free-Free (Bremsstrahlung) Radiation

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We consider the problem of unbound interacting charged particles of very different mass. In the frame of the heavier particle, the light particle is moving at velocity $\mathbf{v}$ with an impact parameter b. The electric field of the heavier particle accelerates the lighter particle, and, as we know, accelerating charges radiate. We suppose that the particles are unbound throughout the problem, hence the name **free-free**. The lighter particle is gradually decelerated because it is losing energy to radiation, hence the name **bremsstrahlung** (German for 'braking').

In an ionised plasma, free-free radiation comes from electrons encountering ions. Encounters between two electrons cannot produce radiation by electric or magnetic dipole processes (they will emit quadrupole radiation, for which the power is lower by $\sim (v/c)^2$). An ion, of course, is also accelerated by the field of a passing electron; however, its mass m_i is much greater than m_e, so its acceleration (and thus its radiated power) is negligible in comparison to the electron's. We assume that the electron's trajectory is a straight line, which will be true of in a typical astrophysical gas (this is essentially the Born approximation). We neglect the effect of radiation reaction on the orbit, generally a good approximation.

There are a few important facts about bremsstrahlung to keep in mind. The process is not an oscillatory one, so the spectrum covers a broad band of frequencies. The total power will be dominated by the closest approach, where the acceleration is largest. The emissivity vs. frequency is nearly flat up to an exponential cutoff at high frequencies. Note that absorption in a plasma can alter the spectrum at low frequencies where the material becomes optically thick (see homework #3).

We first consider the (simpler) case where the electron is moving non-relativistically. The acceleration is

$$\mathbf{a} = \frac{Ze^2}{m_e d^2}\hat{\mathbf{d}} = \frac{Ze^2}{m_e(b^2+v^2t^2)}\hat{\mathbf{d}},$$

where $t=0$ is taken to be the point of closest approach, and d is the distance from the electron to the ion. The dipole electric field at distance r from the electron is

$$|\mathbf{E}| = \frac{ea}{c^2 r}\sin\theta = \frac{Ze^3\sin\theta}{m_e c^2 r(b^2+v^2t^2)}.$$

From the contour integration in assignment #2, the Fourier transform of this electric field is

$$\widetilde{E}_\nu = \int_{-\infty}^{\infty} E(t)e^{2\pi i\nu t}\,dt = \frac{Ze^3\sin\theta}{m_e c^2 r}\int_{-\infty}^{\infty}\frac{e^{2\pi i\nu t}}{b^2+v^2t^2}\,dt = \frac{Ze^3\sin\theta}{m_e c^2 r}\frac{\pi}{bv}e^{-2\pi|\nu|b/v}.$$

The spectrum radiated by a single electron is therefore

$$\frac{dW}{d\nu} = \frac{c}{2\pi}\int\limits_{(4\pi)} r^2\,d\Omega\,|\widetilde{E}_\nu|^2 = \frac{4\pi^2}{3}\frac{Z^2e^6}{m_e^2c^3b^2v^2}e^{-4\pi\nu b/v}.$$

This spectrum is flat for $\nu \ll v/b$ and cuts off exponentially for $\nu \gg v/b$.

If electrons all have the same velocity v, the number of electrons passing through an annulus db of impact parameters around a single ion per unit time is $n_e v 2\pi b\,db$. To get the power radiated per unit volume, we also multiply by the density n_i of ions, yielding

$$\begin{aligned}\frac{dW}{d\nu\,dt\,dV} &= n_e n_i v\int_{b_{min}}^{\infty}\frac{dW}{d\nu}2\pi b\,db \\ &\approx \frac{8\pi^3}{3}\frac{Z^2e^6}{m_e^2c^3v}\int_{b_{min}}^{b_{max}}\frac{db}{b}.\end{aligned}$$

where $dW/d\nu$ is the single-electron spectrum found above. Since this spectrum cuts off exponentially for $\nu > v/b$, we may approximate the exponential by a step function which goes from 1 to 0 at $\nu \sim v/b$; this is equivalent to choosing a maximum impact parameter for the integration $b_{max} \sim v/\nu$ and ignoring the

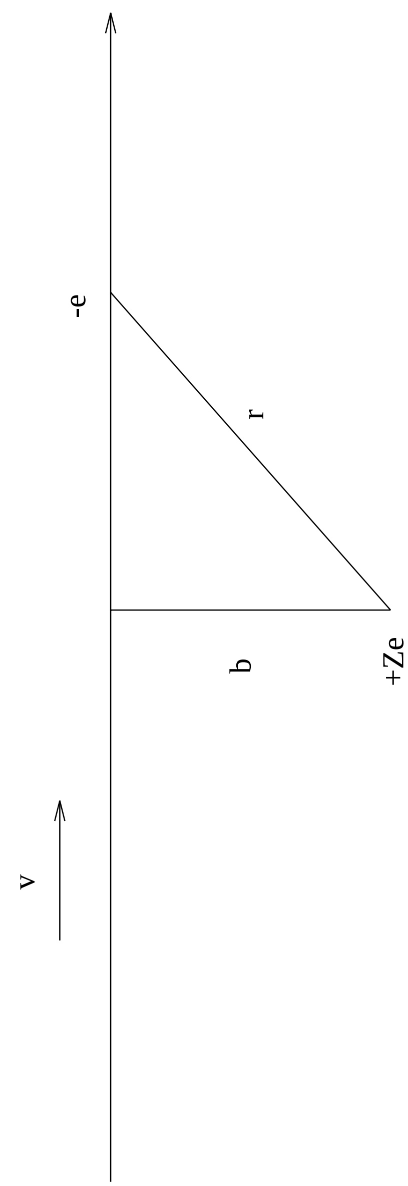

Figure 1: Trajectory of an electron passing an ion.

exponential factor. We put all of our ignorance (both the approximations made thus far and additional quantum mechanical factors) into the Gaunt factor $g_{ff}(v,\nu)$, which is of order unity in most regimes of interest. See Rybicki & Lightman (p. 159–161) for approximations and references to tabulations of the Gaunt factor.

The minimum impact parameter b_{min} will be set by one of the two following considerations:

- The uncertainty principle prohibits the electron from getting any closer than $b_{min} \sim \hbar/p \sim \hbar/m_e v$.
- The small-angle approximation will begin to break down if $b < b_{min}$, where b_{min} is set by $Ze^2/b_{min} \sim m_e v^2$.

The classical criterion will be important when $v \leq Ze^2/\hbar$ ($v/c \leq \alpha Z$ where $\alpha = e^2/\hbar c \approx 1/137$); this will be true for bremsstrahlung from a typical HII region ($T \sim 10^4$ K). At the higher velocities found in the intergalactic gas in galaxy clusters ($T \sim 10^8$ K), the uncertainty principle determines b_{min}.

To make further progress we need to know the velocity distribution of the particles in the gas. Let $dP(\mathbf{v})$ be the probability of a velocity in the range $(\mathbf{v}, \mathbf{v}+d\mathbf{v})$, normalised so that $\int dP(\mathbf{v}) = 1$. To find the emission coefficient from a thermal gas, we simply average over the velocity distribution:

$$\epsilon_\nu = \int_{v_{min}}^{\infty} \frac{dW}{d\nu\, dt\, dV} dP(\mathbf{v}).$$

Note that there is a minimum allowable velocity v_{min} in the problem. This arises because the electron must have a kinetic energy of at least $\frac{1}{2} m_e v^2 = h\nu$ in order to emit a photon of frequency ν. This is a photon discreteness effect; photons must be emitted in quanta.

For Maxwell-Boltzmann statistics, we know that $dP \propto e^{-E/kT} d^3\mathbf{v} = e^{-mv^2/2kT} 4\pi v^2\, dv$, where we have assumed that the velocities are isotropic. For a gas at temperature T,

$$P(v) = 4\pi n_e \left(\frac{m_e}{2\pi kT}\right)^{3/2} v^2 e^{-m_e v^2/2kT}.$$

To do the integral, change variables to $v' = v - v_{min}$. The integration range is then 0 to ∞, and we can pull out a factor $e^{-m_e(-v_{min})^2/kT} = e^{-h\nu/kT}$. The result is

$$\begin{aligned}\epsilon_\nu &= \frac{32\pi}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{Z^2 e^6}{m_e^2 c^3}\left(\frac{m_e}{kT}\right)^{1/2} n_e n_i e^{-h\nu/kT} \bar{g}_{ff} \\ &\approx 6.8\times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff} \text{ erg s}^{-1}\text{ cm}^{-3}\text{ Hz}^{-1},\end{aligned}$$

where $\bar{g}_{ff}$ is the Gaunt factor averaged over velocities. Note where each of the factors in this expression comes from. The exponential cutoff is due to photon discreteness and the exponentially small number of very high-energy electrons in a Maxwellian distribution. The $T^{-1/2}$ arises because the power emitted by a single electron scales as $v^{-1} \propto T^{-1/2}$. The process depends on an electron encountering an ion; the rate of this is obviously proportional to $n_e n_i$.

The total power loss to bremsstrahlung can be found by integrating over all frequencies:

$$\begin{aligned}\frac{dW}{dV\, dt} \int_0^\infty \epsilon_\nu\, d\nu &= \frac{32\pi}{3}\left(\frac{2\pi}{3}\right)^{1/2} \frac{e^6}{hm_e c^3}\left(\frac{kT}{m_e}\right)^{1/2} Z^2 n_e n_i \bar{g} \\ &\approx 1.4\times 10^{-27} Z^2 n_e n_i T^{1/2} \bar{g},\end{aligned}$$

where $\bar{g}$ is now the frequency average of the velocity-averaged Gaunt factor. This factor will be in the range 1.1–1.5, and a good approximation is 1.2. Note that the total power lost to the electron is proportional to $T^{1/2} \propto v \propto E^{1/2}$, where E is the energy of the electron.