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## Chapter 27. Examples for Part VI.

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### 27.1. Example 1. Rough estimates.

Assume that for an order of magnitude estimation the linear approximation gives more or less correct answer up to $\delta \simeq 1$. Estimate the ratio of $\delta(z)$ at $z=99$ and $\delta(z)$ at $z=9$, if it is given that growing and decaying modes are approximately equal to each other at $z=999$.

## Solution

According to the solution given in Section 26.3.

$$
\begin{equation*}
\frac{\delta(99)}{\delta(9)} \simeq \frac{A_{+} 10^{-2}+A_{-} 10^{3}}{A_{+} 10^{-1}+A_{-} 10^{3 / 2}}=\frac{10^{-2}+10^{3} f}{10^{-1}+10^{3 / 2} f} \tag{1}
\end{equation*}
$$

where

$$
\begin{equation*}
f=\frac{A_{-}}{A_{+}} . \tag{2}
\end{equation*}
$$

Taking into account that

$$
\begin{equation*}
A_{+} 10^{-3} \simeq A_{-} 10^{9 / 2} \tag{3}
\end{equation*}
$$

we find that

$$
\begin{equation*}
f \simeq 10^{-3-9 / 2}=10^{-15 / 2} \tag{4}
\end{equation*}
$$

hence

$$
\begin{equation*}
\frac{\delta(99)}{\delta(9)} \simeq \frac{10^{-2}+10^{3-15 / 2}}{10^{-1}+10^{3 / 2-15 / 2}}=\frac{10^{-2}+10^{-9 / 2}}{10^{-1}+10^{-6}} \approx 10^{-2+1}=0.1 \tag{5}
\end{equation*}
$$

### 27.2. Example 2. Amplitude of $\delta$ as a function of time. Alternative derivation.

Derive the equation for the evolution of small density perturbations, $\delta=\left(\rho^{\prime}-\rho\right) / \rho$ after decoupling to show that

$$
\ddot{\delta}+(4 / 3 t) \dot{\delta}-\left(2 / 3 t^{2}\right) \delta=0
$$

## Solution

Starting from

$$
\ddot{R}=-\frac{4 \pi G \rho R}{3}
$$

perturb $R$ and $\rho: R^{\prime}=R(1+h)$, and $\rho^{\prime}=\rho(1+\delta)$.
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Putting this in the perturbed equation

$$
\ddot{R^{\prime}}=-\frac{4 \pi G \rho^{\prime} R^{\prime}}{3}
$$

we obtain

$$
\ddot{R}(1+h)+2 \dot{R} \dot{h}+R \ddot{h}=-\frac{4 \pi G \rho(1+\delta) R(1+h}{3} .
$$

Using unperturbed equation, we obtain linearized equation

$$
\ddot{R} h+2 \dot{R} \dot{h}+R \ddot{h}=-\frac{4 \pi G \rho R(\delta+h)}{3} .
$$

To relate $h$ and $\delta$ we use the conservation of energy equation $\rho R^{3}=\rho R^{3}(1+3 h)(1+\delta)$, or $1=1+3 h+\delta$, so $h=-\delta / 3$.
Thus we have

$$
-\ddot{R} \frac{\delta}{3}-\frac{2}{3} \dot{R}-R \frac{\ddot{\delta}}{3}=-\frac{4 \pi G \rho}{3} R \frac{2}{3} \delta
$$

then

$$
\ddot{\delta}+2 \frac{\dot{R}}{R}+\frac{\ddot{R}}{R} \delta=\frac{8 \pi G \rho}{3} \delta
$$

For the dust-like Universe $R \sim t^{2 / 3}$, so

$$
\frac{\dot{R}}{R}=\frac{2}{3 t}, \quad \frac{\ddot{R}}{R}=\frac{2}{3}\left(\frac{2}{3}-1\right) t^{-2}=-\frac{2}{9 t^{2}} .
$$

From the unperturbed equation

$$
\frac{8 \pi G \rho}{3}=-\frac{2 \ddot{R}}{R}=\frac{4}{9 t^{2}}
$$

Then

$$
\ddot{\delta}+2 \frac{2}{3 t}+\left(-\frac{2}{9}-\frac{4}{9}\right) \frac{\delta}{t^{2}}=0
$$

and finally

$$
\ddot{\delta}+\frac{4}{3 t}-\frac{2}{3 t^{2}} \delta=0 .
$$

Taking trial solution $\delta=A t^{m}$, we obtain

$$
m(m-1)+\frac{4 m}{3}-\frac{2}{3}=0,3 m^{2}+m-2=0
$$

Solutions of this quadratic equation are

$$
m=\frac{-1 \pm \sqrt{1+24}}{6}=\frac{-1 \pm 5}{6}
$$

thus $m_{+}=2 / 3$ and $m_{-}=-1$ (growing and decaying modes). So we have

$$
\delta=A_{+}\left(t / t_{0}\right)^{2 / 3}+A_{-}\left(t / t_{0}\right)^{-1}
$$

