Chapter 26. Linear evolution of the density fluctuations.

CONTENT OF CHAPTER 26

26.1. Equation for fluctuations in the linear approximation.

Let us consider a dust sphere of average density ρ' in a background flat Universe with $k = \Lambda = 0$. The amplitude of the small density perturbation

$$\delta(R) = \frac{\rho'(R) - \rho(R)}{\rho(R)} \tag{1}$$

is a function of scale factor R, where $\rho(R)$ is the average density of the Universe. Starting from

$$\ddot{R} = -\frac{4\pi G\rho R}{3}. \tag{2}$$

Then we can perturb R and ρ:

$$R^{'} = R(1+h, \tag{3}$$

and

$$\rho^{'} = \rho(1+\delta). \tag{4}$$

To relate h and δ we use the conservation of energy equation

$$\rho R^3 = \rho R^3(1+3h)(1+\delta), \tag{5}$$

or

$$1 = 1 + 3h + \delta, \tag{6}$$

hence

$$h = -\delta/3. \tag{7}$$

Now we can present R', $\dot{R}'$ and $\ddot{R}'$ as

$$R' = R(1 - \frac{\delta}{3}), \tag{8}$$

$$\dot{R}' = \dot{R}\frac{dR'}{dR}, \tag{9}$$

$$\ddot{R}' = \ddot{R}\frac{dR'}{dR} + \dot{R}^2\frac{d^2R'}{dR^2}, \tag{10}$$

Putting this in the perturbed equation

$$\ddot{R}' = -\frac{4\pi G\rho' R'}{3}, \tag{11}$$

we have

$$\ddot{R}\frac{d}{dR}[R(1-\frac{\delta}{3})] + \dot{R}^2\frac{d^2}{dR^2}[R(1-\frac{\delta}{3})] = -\frac{4\pi G\rho R}{3}(1+\delta)(1-\frac{\delta}{3}). \tag{12}$$

Taking into account unperturbed equation

$$\ddot{R} = -\frac{4\pi G\rho R}{3} \tag{13}$$

and unperturbed Friedman equation

$$\dot{R}^2 = \frac{8\pi G\rho R^2}{3}, \tag{14}$$

in first order with respect to δ we obtain

$$\frac{4\pi G\rho R}{3}\left\{-\frac{d}{dR}[R(1-\frac{\delta}{3})] + 2R\frac{d^2}{dR^2}[R(1-\frac{\delta}{3})] + 1 + \delta - \frac{\delta}{3}\right\} = 0, \tag{15}$$

thus

$$\delta - R\frac{d\delta}{dR} - \frac{2}{3}R^2\frac{d^2\delta}{dR^2} = 0, \tag{16}$$

Finally, the equation for evolution of $\delta(R)$ can be written in the form

$$\frac{d^2\delta}{dR^2} + \frac{3}{2R}\frac{d\delta}{dR} - \frac{3}{2R^2}\delta = 0. \tag{17}$$

26.2. Solution in the linear approximation.

The general solution of this equation can be represented in terms of two independent modes, one of which is growing, while the other is decaying. Indeed, taking trial solution

$$\delta = AR^m, \tag{18}$$

we obtain

$$m(m-1) + \frac{3m}{2} - \frac{3}{2} = 0, \;\; 2m^2 + m - 3 = 0. \tag{19}$$

Solutions of this quadratic equation are

$$m_{\pm} = \frac{1}{2}\left(-\frac{1}{2} \pm \sqrt{\frac{1}{6} + 4}\right) = \frac{-1 \pm 5}{4}, \tag{20}$$

thus

$$m_{+} = 1 \tag{21}$$

and

$$m_{-} = -\frac{3}{2}. \tag{22}$$

So we have growing and decaying modes:

$$\delta = A_{+}(R/R_0) + A_{-}(R/R_0)^{-3/2}. \tag{23}$$

26.3. Amplitude of fluctuations as a function of redshift .

Taking into account that

$$R/R_0 = (1+z)^{-1}, \tag{24}$$

we can present the $\delta(z)$ as a function of redshift z in the following form

$$\delta = A_{+}(1+z)^{-1} + A_{-}(1+z)^{3/2}. \tag{25}$$