Lecture 6. *Last updated 16.02.10*

VI. CURVATURE OF SPACE-TIME

A. The Riemann curvature tensor

We know that $A_{i,k,l} - A_{i,l,k} = 0$. What can we say about the following commutator $A_{i;\,k;\,l} - A_{i;\,l;\,k}$? Straightforward calculations will show that this is not equal to zero in the presence of gravitational field and can be presented as

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}, \tag{VI.1}$$

where the object R^i_{klm} is obviously a tensor and called the curvature Riemann tensor.
We know that if at least one component of a tensor is not equal to zero in at least one frame of reference, the same is true for any other frame of reference. In other words, tensors (in contrast to the Christoffel symbols) can not be eliminated by transformations of coordinates.
The Riemann tensor describes an actual tidal gravitational field, which is not local and, hence, can not be eliminated even in the locally inertial frame of reference. Let us calculate the curvature Riemann tensor directly:

$$A_{i;k;l} - A_{i;l;k} = A_{i;k,l} - A_{i;l,k} - \Gamma^m_{il} A_{m;k} - \Gamma^m_{kl} A_{i;m} + \Gamma^m_{ik} A_{m;l} + \Gamma^m_{lk} A_{i;m} =$$

$$= \left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} - \Gamma^m_{il}\left(A_{m,k} - \Gamma^p_{mk} A_p\right) + \Gamma^m_{ik}\left(A_{m,l} - \Gamma^p_{ml} A_p\right) =$$

$$= A_{i,k,l} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{ik,l} A_m - A_{i,l,k} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{il,k} A_m - \Gamma^m_{il} A_{m,k} + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{ik} A_{m,l} - \Gamma^m_{ik}\Gamma^p_{ml} A_p =$$

$$= -\Gamma^m_{ik,l} A_m + \Gamma^m_{il,k} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p - \Gamma^m_{ik}\Gamma^p_{ml} A_p = -\Gamma^m_{ik,l} A_m + \Gamma^m_{il,k} A_m + \Gamma^p_{il}\Gamma^m_{pk} A_m - \Gamma^p_{ik}\Gamma^m_{pl} A_m =$$

$$= \left(-\Gamma^m_{ik,l} + \Gamma^m_{il,k} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl}\right) A_m = R^m_{ikl} A_m. \tag{VI.2}$$

Finally

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl}. \tag{VI.3}$$

Similar equations can be written for tensors of higher ranks, for example

$$A_{ik;\,l;\,m} - A_{ik;\,m;\,l} = A_{in} R^n_{klm} + A_{nk} R^n_{ilm}. \tag{VI.4}$$

Let us introduce the covariant presentation of the Riemann tensor:

$$R_{iklm} = g_{in} R^n_{klm}. \tag{VI.5}$$

By straightforward calculations one can show that

$$R_{iklm} = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}). \tag{VI.6}$$

B. Symmetry properties of the Riemann tensor

There are several symmetry properties of the curvature tensor:
1) The Riemann tensor is antisymmetric with respect to permutations of indices within each pair

$$R_{iklm} = -R_{kilm} = -R_{ikml}. \tag{VI.7}$$

2) The Riemann tensor is symmetric with respect to permutations of pairs of indices

$$R_{iklm} = R_{lmik}. \tag{VI.8}$$

3) The cyclic sum formed by permutation of any three indices is equal to zero

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{VI.9}$$

C. Bianchi Identity

The most important property of the Riemann tensor is so called the Bianchi identity:

$$R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = 0. \tag{VI.10}$$

It is easy to verify this identity in a locally inertial frame of reference, where

$$\Gamma^i_{kl} = 0, \tag{VI.11}$$

hence

$$R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = \tag{VI.12}$$

$$\Gamma^n_{il,m,k} - \Gamma^n_{ik,m,l} + \Gamma^n_{ik,l,m} - \Gamma^n_{im,l,k} + \Gamma^n_{im,k,l} - \Gamma^n_{il,k,m} = 0. \tag{VI.13}$$

Taking into account that the Bianchi identity is of a tensor character, we can conclude that it valid in any other frame of reference.

D. The Ricci tensor and the scalar curvature

Now we can introduce a second rank curvature tensor, called the Ricci tensor, as follows

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk}. \tag{VI.14}$$

We can also introduce a zero rank curvature tensor, i.e. a scalar, called the scalar curvature:

$$R = g^{ik} R_{ik}. \tag{VI.15}$$

1. *The important consequence of Bianchi identity*

After contracting the Biancci identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0 \quad \text{(VI.16)}$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0. \quad \text{(VI.17)}$$

According to the definition of Ricci tensor (VI.14), the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}. \quad \text{(VI.18)}$$

Taking into account that the Riemann tensor is antisymmetric with respect to permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km}, \quad \text{(VI.19)}$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}. \quad \text{(VI.20)}$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i}, \quad \text{(VI.21)}$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}. \quad \text{(VI.22)}$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0. \quad \text{(VI.23)}$$

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km} g^{ip} R_{kpml;i} + g^{km} R_{kl;m} - g^{km} R_{km;l} = \left(g^{km} g^{ip} R_{kpml}\right)_{;i} + \left(g^{km} R_{kl}\right)_{;m} - \left(g^{km} R_{km}\right)_{;l} = 0. \quad \text{(VI.24)}$$

In the first term expression in brackets can be simplified as

$$g^{km} g^{ip} R_{kpml} = g^{ip} R_{pl} = R^i_l. \quad \text{(VI.25)}$$

In the second term the expression in brackets can be simplified as

$$g^{km} R_{kl} = R^m_l. \quad \text{(VI.26)}$$

According to the definition of the scalar curvature (VI.15), the third term can be simplified as

$$\left(g^{km} R_{km}\right)_{;l} = R_{;l} = R_{,l}. \quad \text{(VI.27)}$$

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0, \quad \text{(VI.28)}$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \ \text{ or } \ R^i_{l;i} - \frac{1}{2} R_{,l} = 0. \quad \text{(VI.29)}$$

Thus the important consequence of Bianchi identity is

$$R^i_{l;i} - \frac{1}{2} R_{,l} = 0. \quad \text{(VI.30)}$$

E. Geodesic deviation equation

The geodesic deviation equation is an equation involving the Riemann curvature tensor, which measures the change in separation of neighboring geodesics. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics. Let the 4-velocity along one geodesic is

$$u^i = \frac{dx^i}{ds}. \tag{VI.31}$$

There is an infinitesimal separation vector between the two geodesics η^i. Then the relative acceleration, a^i, is

$$a^i = \frac{d^2\eta^i}{ds^2}. \tag{VI.32}$$

It is possible to show that

$$a^i = R^i_{klm} u^k u^l \eta^m. \tag{VI.33}$$

If gravitational field is weak and all motions are slow

$$u^i \approx \delta^i_0, \tag{VI.34}$$

and the above equation is reduced to the Newtonian equation for the tidal acceleration.

F. Stress-Energy Tensor

The stress-energy tensor (sometimes stress-energy-momentum tensor), T_{ik}, describes the density and flux of energy and momentum.
In general relativity this tensor is symmetric and contains ten independent components:
The component T_{00} represents the energy density (1 component).
The components $T_{0\alpha}$ ($\alpha = 1, 2, 3$) represent the flux of energy across the surface which is normal to the x_α-axis. These components are equivalent to the components $T_{\alpha 0}$ which describe the density of the α^{th} momentum (3 components).
The components $T_{\alpha\beta}$ (α, $\beta = 1, 2, 3$) represent flux of α^{th} momentum across the surface which is normal to the x_β-axis.
In particular, the diagonal components $T_{\alpha alpha}$ represents a pressure-like quantity, normal stress (3 components).
Non-diagonal components $T_{\alpha\beta}$ ($\alpha \neq \beta$), represent shear stress (3 components).
All these ten components participate in the generation of a gravitational field, while in Newton gravity the only source of gravitational field is the mass density.

1. *Conservation of energy-momentum in gravitational field*

According to physics in absence of gravitational field the stress-energy tensor satisfies the following conservation law:

$$T^i_{k,i} = 0. \tag{VI.35}$$

We know from previous lectures that according ", $\rightarrow$;-rule" in the presence of gravitational field this should be rewritten as

$$T^i_{k;i} = 0. \tag{VI.36}$$

G. Heuristic "Derivation" of EFEs

It seems like a good idea to relate the Ricci tensorto the stress-energy tensor.
The most general form of the second rank tensor formed from the metric tensor g_{ik} and containing second derivatives of the metric tensor g_{ik}, let us call it the Einstein tensor, is

$$G_{ik} = R_{ik} + \alpha g_{ik} R. \tag{VI.37}$$

As follows from the the previous section

$$G^i_{k;i} = (g^{in} G_{nk})_{;i} = R^i_{k;i} + \alpha \delta^i_k R_{,i} = (\frac{1}{2} + \alpha) R_{,k}. \tag{VI.38}$$

Let us assume that the EFEs have the following form

$$G_{ik} = \kappa T_{ik}, \tag{VI.39}$$

where the constant κ is called the Einstein constant. Multiplying this by g^{mk} we obtain

$$R^m_i + \alpha \delta^m_i R = \kappa T^m_k. \tag{VI.40}$$

Taking covariant divergence of LHS and RHS of this equation we obtain

$$(\alpha + \frac{1}{2}) R_{;k} = \kappa T^m_{k;\,m} = 0, \tag{VI.41}$$

hence

$$\alpha = -\frac{1}{2}, \tag{VI.42}$$

and final EFEs are

$$R^i_k - \frac{1}{2} \delta^i_k R = \kappa T^i_k. \tag{VI.43}$$

To determine κ we can use the so called the correspondence principle, which says that the EFEs in weak-field and the slow-motion approximation should be reduced to Newton's law of gravity, i.e. to the Poisson's equation

$$\Delta \phi = 4\pi G \rho. \tag{VI.44}$$

By straightforward calculations one can prove that such reduction is possible only if

$$\kappa = \frac{8\pi G}{c^4}. \tag{VI.45}$$

Finally, EFEs can be written as

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}. \tag{VI.46}$$

Despite the simple appearance of this equation it is, in fact, quite complicated. Given a specified distribution of matter and energy in the form of a stress-energy tensor, the EFE are understood to be equations for the metric tensor g_{ik}, as both the Ricci tensor and Ricci scalar depend on the metric (in a complicated nonlinear manner). In fact, when fully written out, the EFEs are the system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations. In other words, Despite the simple appearance of the EFEs they are, in fact, rather complicated.
Solutions of the Einstein field equations model an extremely wide variety of gravitational fields.
Some of them are really exotic, for example the solution corresponding to the so called **wormhole** [A wormhole is a hypothetical topological feature of spacetime that is essentially a 'shortcut' through space and time. A wormhole has at least two mouths which are connected to a single throat. If the wormhole is traversable, matter can 'travel' from one mouth to the other by passing through the throat. While there is no observational evidence for wormholes, spacetimes containing wormholes are known to be valid solutions of the Einsteins equations.
Gravitational waves and **black holes** are also solutions of EFEs.

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