Lecture 1

I. INTRODUCTION

A. About this course

This course is an introduction to General Relativity (GR) and includes:

Explanation of the fundamental principles of GR.
The motion of particles in a given gravitational field.
The propagation of electromagnetic waves in a gravitational field.
The derivation of Einstein's field equations from the basic principles.
The derivation of the Schwarzschild solution.
Analysis of the Kerr solution.
A discussion of physical aspects of strong gravitational fields around black holes.
The generation, propagation and detection of gravitational waves.
The weak general relativistic effects in the Solar System and binary pulsars.
The experimental tests of General Relativity.

B. The principle of equivalence

The basic postulate of the GR states that a uniform gravitational field is equivalent to (which means is not distinguishable from) a uniform acceleration. In practice this means that a person cannot feel (locally) the difference between standing on the surface of some gravitating body (for example the Earth) and moving in a rocket with corresponding acceleration (**Fig. 1.1**). According to Einstein (**Fig.1.2**) these effects are actually the same.

The important consequence of the equivalence principle is that any gravitational field can be eliminated in free falling frames of references, which are called local inertial frames or local galilean frames **Fig. 1.3**). In other words, there is no experiment to distinguish between being weightless far out from gravitating bodies in space and being in free-fall in a gravitational field. Another illustration of this principle is shown on **Fig.1.4**. This picture, as well as some other images, is taken from the very interesting astronomical website by Nick Strobel.

The Principle of Equivalence in Newtonian Gravity. All bodies in a given gravitational field will move in the same manner, if initial conditions are the same. In other words, in given gravitational field all bodies move with the same acceleration. In absence of gravitational field, all bodies move also with the same acceleration relative to the non-inertial frame. Thus we can formulate the Principle of Equivalence which says: locally, any non-inertial frame of reference is equivalent to a certain gravitational field.

Globally (not locally), "actual" gravitational fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational Fields generated by gravitating bodies decay with distance.
In Newton's theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi, \tag{I.1}$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass, which is the gravitational analog of the electric charge in the theory of electromagnetism. The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1, \tag{I.2}$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

The Principle of Equivalence in GR. As it is known from every course on Special Relativity (SR), this theory works only in the frames of reference of the special kind called Global Inertial Frames of Reference. For such frames of reference the following combination of time and space coordinates remains invariant in all global inertial frames of references

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{I.3}$$

This combination is called the interval. All space-time coordinates in different global inertial frames of reference are related to each other by the Lorentz transformations. It is also known that these transformations leave the shape of the interval unchanged. But this is not the case if one considers transformation of coordinates in more general case, when at least one of frames of reference is non-inertial. This interval is not reduced anymore to the simple sum of squares of the coordinate differentials and can be written in the following more general quadratic form:

$$ds^2 = g_{ik}dx^i dx^k \equiv \sum_{i=0}^{3}\sum_{k=0}^{3} g_{ik}dx^i dx^k, \tag{I.4}$$

where repeating indices mean summation. In inertial frames of reference

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \ \text{ and } \ g_{ik} = 0, \ \text{ if } \ i \neq k. \tag{I.5}$$

Example. Transformation to an uniformly rotating frame:

$$x = x' \cos \Omega t - y' \sin \Omega t, \quad y = x' \sin \Omega t + y' \cos \Omega t, \quad z = z', \tag{I.6}$$

where Ω is the angular velocity of rotation around z-axis. In this non-inertial frame of reference as one can see by straightforward calculations

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y'dx'dt - 2\Omega x'dy'dt. \tag{I.7}$$

C. Gravity as a space-time geometry

The fundamental physical concept of GR is that a gravitational field is identical to geometry of curved space-time. This idea, called the Geometrical Principle, entirely determines the mathematical structure of General Relativity. According to the GR gravity is nothing but a manifestation of space-time 4-geometry, this geometry is determined by by metric

$$ds^2 = g_{ik}(x^m)dx^i dx^k, \tag{I.8}$$

where $g_{ik}(x^m)$ is called the metric tensor (what exactly is meant by the term "tensor" we will discuss in the next lecture). At the present moment we can consider $g_{ik}(x^m)$ as a 4×4 -matrix and all its components in a general case can depend on all 4 coordinates x^m, where $m = 0, 1, 2, 3$. All information about the geometry of space-time is contained in $g_{ik}(x^m)$. The dependence of $g_{ik}(x^m)$ on x^m means that this geometry is different in different events, which implies that the space-time is curved and its geometry is not Euclidian. Such sort of geometry is the the subject of mathematical discipline called Differential Geometry developed in XIX Century. Examples of highly curved space-time are shown on **Fig.1.5** and **Fig.1.6**.

The GR gives a very simple and natural explanation of the Principle of Equivalence: in curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in a flat space-time when free particles move along straight lines which are geodesics in flat space-time. What is the geodesic we will discuss in the next lectures.

If we know g_{ik}, we can determine completely the motion of test particles and the performance of all test fields. This is one of the main statements of GR. [When we say test particle or test field we mean that gravitational field generated by these test objects is negligible.] In the next lectures we will see that the metric tensor g_{ik} itself, and hence geometry, is determined by physical content of the space-time.

In any curved space-time (i.e in the actual gravitational field) there is no global galilean frames of reference. In flat space-time, if me work in non-inertial frames of reference metrics looks like the metric in gravitational field (because according to the Equivalence Principle, locally, actual gravitational field is not distinguishable from corresponding non-inertial frame of reference), nevertheless local (not global) galilean frames of reference do exist.

The local galilean frame of reference is equivalent to the freely falling frame of reference in which locally gravitational field is eliminated. From geometrical point of view to eliminate gravitational field locally means to find such frame of reference in which

$$g_{ik} \rightarrow \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{I.9}$$

D. The principle of covariance

If space-time is flat and one works with inertial frames of reference then the world lines of free particles are straight lines. For particles moving with acceleration the world lines are curved (see **Fig.1.7**).
The fact that all bodies move with the same acceleration in a given gravitational field means that this gravitational field is really a manifestation of properties of space-time itself and that there is no way experimentally to discriminate between a gravitational field and non-inertial frame of reference. More mathematically this statement can be formulated as the Principle of Covariance which says: the shape of all physical equations should be the same in an arbitrary frame of reference. Otherwise the physical equations [being different in gravitational field and in inertial frames of reference] would have different solutions, in other words, these equations would predict the difference between a gravitational field and a non-inertial frame of reference and ,hence, would contradict to the experimental data. This principle refers to the most general case of non-inertial frames (in contrast to the SR which works only in inertial frames of reference).