Relativity and Gravitation (MTH720U/MTHM033) 2010

LECTURE NOTES

Last updated 07.03.10

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1. *Content of lecture notes*

2. *Course information*

About the course

This course is an introduction to General Relativity and includes:
Explanation of the fundamental principles of GR. The motion of particles in a given gravitational field. The propagation of electromagnetic waves in a gravitational field. The derivation of Einstein's field equations from the basic principles. The derivation of the Schwarzschild solution. Analysis of the Kerr solution. A discussion of physical aspects of strong gravitational fields around black holes. The generation, propagation and detection of gravitational waves. The weak general relativistic effects in the Solar System and binary pulsars. The experimental tests of General Relativity.

Assessment

Course-work 0%, exam 100%

Key Objectives

1. Effects of General Relativity in the Solar System and in the Universe:

you should have a good understanding of the importance of general relativity in physics and astronomy.

2. Curvilinear Coordinates, Covariant Differentiation:

You should be able to operate with concepts of differential geometry and understand the deep relationship between physics and geometry.

3. Motion of Particles in a Gravitational Field:

You should understand the fundamental difference in the motion of particles in relativistic theory of gravitation and in Newtonian theory. You should be able to write down and solve in the simplest cases the geodesic equation.

4. The Curvature Tensor and the Einstein Equations:

You should understand basic physical principle of the least action and have good qualitative understanding of the most important stages of the derivation of these equations.

5. Black Holes:

You should understand what is event horizon and what is the limit of stationarity. You should be able to describe the main effects of strong gravitational field around black hole and have idea how the black holes could be discovered.

6. Gravitational Waves:

You should be able to derive the wave equation for propagation of gravitational radiation, understand why gravitational waves are transverse and traceless, what is similarity and what is the difference with electromagnetic waves. You should also be able to produce order of magnitude estimations of amplitudes of gravitational waves from astrophysical sources of gravitational radiation.

Lecture 1. *Last updated 07.03.10*

I. INTRODUCTION

A. About this course

This course is an introduction to General Relativity (GR) and includes:
Explanation of the fundamental principles of GR.
The motion of particles in a given gravitational field.
The propagation of electromagnetic waves in a gravitational field.
The derivation of Einstein's field equations from the basic principles.
The derivation of the Schwarzschild solution.
Analysis of the Kerr solution.
A discussion of physical aspects of strong gravitational fields around black holes.
The generation, propagation and detection of gravitational waves.
The weak general relativistic effects in the Solar System and binary pulsars.
The experimental tests of General Relativity.

B. The principle of equivalence

The basic postulate of the GR states that a uniform gravitational field is equivalent to (which means is not distinguishable from) a uniform acceleration. In practice this means that a person cannot feel (locally) the difference between standing on the surface of some gravitating body (for example the Earth) and moving in a rocket with corresponding acceleration (**Fig. 1.1**).
According to Einstein (**Fig.1.2**) these effects are actually the same.
The important consequence of the equivalence principle is that any gravitational field can be eliminated in free falling frames of references, which are called local inertial frames or local galilean frames **Fig. 1.3**).
In other words, there is no experiment to distinguish between being weightless far out from gravitating bodies in space and being in free-fall in a gravitational field. Another illustration of this principle is shown on **Fig.1.4**. This picture, as well as some other images, is taken from the very interesting astronomical website by Nick Strobel.

1. *The Principle of Equivalence in Newtonian Gravity.*

All bodies in a given gravitational field will move in the same manner, if initial conditions are the same. In other words, in given gravitational field all bodies move with the same acceleration. In absence of gravitational field, all bodies move also with the same acceleration relative to the non-inertial frame. Thus we can formulate the Principle of Equivalence which says: locally, any non-inertial frame of reference is equivalent to a certain gravitational field.

Globally (not locally), "actual" gravitational fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational Fields generated by gravitating bodies decay with distance.
In Newton's theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi, \tag{I.1}$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass, which is the gravitational analog of the electric charge in the theory of electromagnetism. The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1, \tag{I.2}$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

2. *The Principle of Equivalence in GR.*

As it is known from every course on Special Relativity (SR), this theory works only in the frames of reference of the special kind called Global Inertial Frames of Reference. For such frames of reference the following combination of time and space coordinates remains invariant in all global inertial frames of references

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2. \tag{I.3}$$

This combination is called the interval. All space-time coordinates in different global inertial frames of reference are related to each other by the Lorentz transformations. It is also known that these transformations leave the shape of the interval unchanged. But this is not the case if one considers transformation of coordinates in more general case, when at least one of frames of reference is non-inertial. This interval is not reduced anymore to the simple sum of squares of the coordinate differentials and can be written in the following more general quadratic form:

$$ds^2 = g_{ik}dx^i dx^k \equiv \sum_{i=0}^{3}\sum_{k=0}^{3} g_{ik}dx^i dx^k, \tag{I.4}$$

where repeating indices mean summation. In inertial frames of reference

$$g_{00} = 1, \quad g_{11} = g_{22} = g_{33} = -1, \text{ and } g_{ik} = 0, \text{ if } i \neq k. \tag{I.5}$$

3. *Example.*

Transformation to an uniformly rotating frame is

$$x = x'\cos\Omega t - y'\sin\Omega t, \;\; y = x'\sin\Omega t + y'\cos\Omega t, \;\; z = z', \tag{I.6}$$

where Ω is the angular velocity of rotation around z-axis. In this non-inertial frame of reference as one can see by straightforward calculations

$$ds^2 = [c^2 - \Omega^2(x'^2 + y'^2)]dt^2 - dx'^2 - dy'^2 - dz'^2 + 2\Omega y'dx'dt - 2\Omega x'dy'dt. \tag{I.7}$$

C. Gravity as a space-time geometry

The fundamental physical concept of GR is that a gravitational field is identical to geometry of curved space-time. This idea, called the Geometrical Principle, entirely determines the mathematical structure of General Relativity. According to the GR gravity is nothing but a manifestation of space-time 4-geometry, this geometry is determined by by metric

$$ds^2 = g_{ik}(x^m)dx^i dx^k, \tag{I.8}$$

where $g_{ik}(x^m)$ is called the metric tensor (what exactly is meant by the term "tensor" we will discuss in the next lecture). At the present moment we can consider $g_{ik}(x^m)$ as a 4×4 -matrix and all its components in a general case can depend on all 4 coordinates x^m, where $m = 0,\ 1,\ 2,\ 3$. All information about the geometry of space-time is contained in $g_{ik}(x^m)$. The dependence of $g_{ik}(x^m)$ on x^m means that this geometry is different in different events, which implies that the space-time is curved and its geometry is not Euclidian. Such sort of geometry is the the subject of mathematical discipline called Differential Geometry developed in XIX Century. Examples of highly curved space-time are shown on **Fig.1.5** and **Fig.1.6**.

The GR gives a very simple and natural explanation of the Principle of Equivalence: in curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in a flat space-time when free particles move along straight lines which are geodesics in flat space-time. What is the geodesic we will discuss in the next lectures.

If we know g_{ik}, we can determine completely the motion of test particles and the performance of all test fields. This is one of the main statements of GR. [When we say test particle or test field we mean that gravitational field generated by these test objects is negligible.] In the next lectures we will see that the metric tensor g_{ik} itself, and hence geometry, is determined by physical content of the space-time.

In any curved space-time (i.e in the actual gravitational field) there is no global galilean frames of reference. In flat space-time, if me work in non-inertial frames of reference metrics looks like the metric in gravitational field (because according to the Equivalence Principle, locally, actual gravitational field is not distinguishable from corresponding non-inertial frame of reference), nevertheless local (not global) galilean frames of reference do exist.

The local galilean frame of reference is equivalent to the freely falling frame of reference in which locally gravitational field is eliminated. From geometrical point of view to eliminate gravitational field locally means to find such frame of reference in which

$$g_{ik} \to \eta_{ik} \equiv \mathrm{diag}(1, -1, -1, -1). \tag{I.9}$$

D. The principle of covariance

If space-time is flat and one works with inertial frames of reference then the world lines of free particles are straight lines. For particles moving with acceleration the world lines are curved (see **Fig.1.7**).

The fact that all bodies move with the same acceleration in a given gravitational field means that this gravitational field is really a manifestation of properties of space-time itself and that there is no way experimentally to discriminate between a gravitational field and non-inertial frame of reference. More mathematically this statement can be formulated as the Principle of Covariance which says: the shape of all physical equations should be the same in an arbitrary frame of reference. Otherwise the physical equations [being different in gravitational field and in inertial frames of reference] would have different solutions, in other words, these equations would predict the difference between a gravitational field and a non-inertial frame of reference and ,hence, would contradict to the experimental data. This principle refers to the most general case of non-inertial frames (in contrast to the SR which works only in inertial frames of reference).

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Lecture 2. *Last updated 07.03.10*

II. TENSORS

A. The principle of covariance and tensors

The Principle of Covariance predetermines the mathematical structure of General Relativity: all equations should contain tensors only. By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another. Taking into account that non-inertial frames of reference in the 4-dimensional space-time correspond to curvilinear coordinates, it is necessary to develop four-dimensional differential geometry in arbitrary curvilinear coordinates.

B. Transformation of coordinates

Let us consider the transformation of coordinates from one frame of reference $(x^0,\, x^1,\, x^2,\, x^3)$ to another, $({x'}^0,\, {x'}^1,\, {x'}^2,\, {x'}^3)$:

$$x^0 = f^0(x'^0, x'^1, x'^2, x'^3),\;\; x^1 = f^1(x'^0, x'^1, x'^2, x'^3),\;\; x^2 = f^2(x'^0, x'^1, x'^2, x'^3),\;\; x^3 = f^3(x'^0, x'^1, x'^2, x'^3). \tag{II.1}$$

Then

$$dx^i = \frac{\partial x^i}{\partial x'^k}dx'^k = S^i_k dx'^k,\;\; i,k = 0,\ 1,\ 2,\ 3,\;\; \text{where}\;\; S^i_k = \frac{\partial x^i}{\partial x'^k} \tag{II.2}$$

is a transformation matrix. Remember that all repeating indices mean summation, otherwise even such a basic transformation would be very ugly when written. To demonstrate that summation convention is really very useful, I will write, the first and the last time, the same transformation without using the summation convention:

$$\begin{cases} dx^0 = \frac{\partial x^0}{\partial x'^0}dx'^0 + \frac{\partial x^0}{\partial x'^1}dx'^1 + \frac{\partial x^0}{\partial x'^2}dx'^2 + \frac{\partial x^0}{\partial x'^3}dx'^3 = S^0_0 dx'^0 + S^0_1 dx'^1 + S^0_2 dx'^2 + S^0_3 dx'^3, \\ dx^1 = \frac{\partial x^1}{\partial x'^0}dx'^0 + \frac{\partial x^1}{\partial x'^1}dx'^1 + \frac{\partial x^1}{\partial x'^2}dx'^2 + \frac{\partial x^1}{\partial x'^3}dx'^3 = S^1_0 dx'^0 + S^1_1 dx'^1 + S^1_2 dx'^2 + S^1_3 dx'^3, \\ dx^2 = \frac{\partial x^2}{\partial x'^0}dx'^0 + \frac{\partial x^2}{\partial x'^1}dx'^1 + \frac{\partial x^2}{\partial x'^2}dx'^2 + \frac{\partial x^2}{\partial x'^3}dx'^3 = S^2_0 dx'^0 + S^2_1 dx'^1 + S^2_2 dx'^2 + S^2_3 dx'^3, \\ dx^3 = \frac{\partial x^3}{\partial x'^0}dx'^0 + \frac{\partial x^3}{\partial x'^1}dx'^1 + \frac{\partial x^3}{\partial x'^2}dx'^2 + \frac{\partial x^3}{\partial x'^3}dx'^3 = S^3_0 dx'^0 + S^3_1 dx'^1 + S^3_2 dx'^2 + S^3_3 dx'^3 \end{cases} \tag{II.3}$$

C. Contravariant and covariant tensors

Now we can give the definition of the Contravariant four-vector: The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = S^i_k A'^k. \tag{II.4}$$

Let φ is scalar field, then

$$\frac{\partial \varphi}{\partial x^i} = \frac{\partial \varphi}{\partial x'^k}\frac{\partial x'^k}{\partial x^i} = \tilde{S}_i^k \frac{\partial \varphi}{\partial x'^k}, \tag{II.5}$$

where $\tilde{S}_i^k$ is another transformation matric. What is the relation of this matrix to the previous transformation matrix S_k^i? If we take product of these matrices, we obtain

$$S_n^i \tilde{S}_k^n = \frac{\partial x^i}{\partial x'^n}\frac{\partial x'^n}{\partial x^k} = \frac{\partial x^i}{\partial x^k} = \delta_k^i, \tag{II.6}$$

where δ_k^i is so called Kronneker symbol, which actually is nothing but the unit matrix:

$$\delta_k^i = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{II.7}$$

In other words $\tilde{S}_k^i$ is inverse or reciprocal with respect to S_k^i.
Now we can give the definition of the Covariant four-vector: The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k. \tag{II.8}$$

Note, that for contravariant vectors we always use upper indices, which are called contravariant indices, while for covariant vectors we use low indices, which are called covariant indices. In General Relativity summation convention always means that one of two repeating indices should be contravariant and another should be covariant.For example,

$$A^i B_i = A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{II.9}$$

is the scalar product.
There is no summation if both indices are, say, covariant, for example:

$$A_i B_i = \begin{array}{l} A_0 B_0, \text{ if } i = 0, \\ A_1 B_1, \text{ if } i = 1, \\ A_2 B_2, \text{ if } i = 2, \\ A_3 B_3, \text{ if } i = 3. \end{array} \tag{II.10}$$

Now we can generalize the definitions of vectors and introduce tensors entirely in terms of transformation laws.
Scalar, A, is the tensor of the 0 rank. It has only $4^0 = 1$ component and 0 number of indices. Transformation law is

$$A = A', \tag{II.11}$$

we see that transformation matrices appear in transformation law 0 times.
Contravariant and covariant vectors are tensors of the 1 rank. They have $4^1 = 4$ components and 1 index. Corresponding transformation laws are

$$A^i = S_n^i A'^n, \tag{II.12}$$

$$A_i = \tilde{S}_i^n A'_n, \tag{II.13}$$

we see only 1 transformation matrix in each transformation law.

Contravariant tensor of the 2 rank has $4^2 = 16$ components and 2 contravariant indices. Corresponding transformation law is

$$A^{ik} = S^i_n S^k_m A'^{nm}, \tag{II.14}$$

we see 2 transformation matrices in the transformation law.
Covariant tensor of the 2 rank has $4^2 = 16$ components and 2 covariant indices. Corresponding transformation law is

$$A_{ik} = \tilde{S}^n_i \tilde{S}^m_k A'_{nm}, \tag{II.15}$$

we see 2 transformation matrices in the transformation law.
Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{II.16}$$

we see 2 transformation matrices in the transformation law.
Covariant tensor of the 3 rank has $4^3 = 64$ components and 3 covariant indices. Corresponding transformation law is...and so on.
The most general definition: Mixed tensor of the $N + M$ rank with N contravariant and M covariant indices, has $4^{N+M} = 2^{2(N+M)}$ components and $N + M$ indices. Corresponding transformation law is

$$A^{i_1\, i_2\, \ldots\, i_N}_{k_1\, k_2\, \ldots\, k_M} = S^{i_1}_{n_1} S^{i_2}_{n_2} \ldots S^{i_N}_{n_N} \tilde{S}^{m_1}_{k_1} \tilde{S}^{m_2}_{k_2} \ldots \tilde{S}^{m_M}_{k_M} A'^{n_1\, n_2\, \ldots\, n_N}_{m_1 m_2 \ldots m_M}, \tag{II.17}$$

we see $N + M$ transformation matrices in the transformation law.

D. Reciprocal tensors

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik} B^{kl} = \delta^l_i. \tag{II.18}$$

Now we can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik} g^{kl} = \delta^l_i. \tag{II.19}$$

With the help of the metric tensor and its reciprocal we can form contravariant tensors from covariant tensors and vice versa, for example:

$$A^i = g^{ik} A_k, \quad A_i = g_{ik} A^k, \tag{II.20}$$

in other words we can rise and descend indices as we like, like a kind of juggling with indices. We can say that contravariant, covariant and mixed tensors can be considered as different representations of the same geometrical object.
For the contravariant metric tensor itself we have very important representation in terms of the transformation matrix from locally inertial frame of reference (galilean frame) to an arbitrary non-inertial frame, let us denote it as $S^i_{(0)k}$. We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1), \tag{II.21}$$

hence

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{lm} = S^i_{(0)0} S^k_{(0)0} - S^i_{(0)1} S^k_{(0)1} - S^i_{(0)2} S^k_{(0)2} - S^i_{(0)3} S^k_{(0)3}. \tag{II.22}$$

This means that if we know the transformation law from the local galilean frame of reference to an arbitrary frame of reference, we know the metric at this arbitrary frame of reference and, hence, we know the gravitational field which is identical to geometry!

E. Examples

Problem: Given that g_{ik} is a covariant tensor of the second rank and that

$$ds^2 = g_{ik}dx^i dx^k, \tag{II.23}$$

prove that ds is a scalar.

Solution:

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2, \tag{II.24}$$

hence $ds = ds'$ which means that ds is a scalar.

Problem: How many independent components in the metric tensor?

Solution: First, let us prove that the metric tensor is symmetric, i.e.

$$g_{ik} = g_{ki}. \tag{II.25}$$

Indeed,

$$ds^2 = g_{ik}dx^i dx^k = \frac{1}{2}(g_{ik}dx^i dx^k + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki}dx^k dx^i + g_{ik}dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik})dx^i dx^k =$$

$$= \tilde{g}_{ik}dx^i dx^k, \tag{II.26}$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}), \tag{II.27}$$

which is obviously a symmetric one. Then we just drop " ˜ ". The end of proof. Now the answer is obvious: altogether we have 4×4 components, 4 components on the diagonal, $3 + 2 + 1 = 6$ components above the diagonal and $3+2+1 = 6$ components under the diagonal and we know that these components are equal to components above the diagonal. Thus the final answer is there are $4 + 6 = 10$ independent components.

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Lecture 3. *Last updated 07.03.10*

III. PHYSICAL GEOMETRY OF SPACE-TIME

A. Proper time

One of the most central problems in the geometry of 4-spacetime can be formulated as follows. If the metric tensor is given, how is actual (measurable) time and distances related with coordinates x^0, x^1, x^2, x^3 chosen in arbitrary way? Let us consider the world line of an observer who uses some clock to measure the actual or proper time, $d\tau$, between two infinitesimally close events in the same place in space. How $d\tau$ is related to coordinate time dx^0. Obviously we should put in the interval

$$dx^1 = dx^2 = dx^3. \tag{III.1}$$

Let us define proper time exactly as in Special Relativity:

$$d\tau = \frac{ds}{c}, \tag{III.2}$$

then we have

$$ds^2 \equiv c^2 d\tau^2 = g_{ik} dx^i dx^k = g_{00}(dx^0)^2, \tag{III.3}$$

thus

$$d\tau = \frac{1}{c}\sqrt{g_{00}}dx^0. \tag{III.4}$$

For the proper time between any two events which are not necessary infinitesimally close occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0. \tag{III.5}$$

B. Physical distance

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2. \tag{III.6}$$

To define dl we will use a light signal according to the following procedure:
From some point B with spatial coordinates $x^\alpha + dx^\alpha$ a light signal emitted at the moment corresponding to time coordinate $x^0 + dx^{0(1)}$ propagates to a point A with spatial coordinates x^α. Then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected at the point B at the moment corresponding to time coordinate $x^0 + dx^{0(2)}$ (see Fig.3.1).

According to both Special and General Relativity the interval between any two events which belong to the same world line of light is always equal to zero:

$$ds = 0. \tag{III.7}$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = \frac{1}{g_{00}} \left(-g_{0\alpha} dx^\alpha - \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \right)$$

$$dx^{0(2)} = \frac{1}{g_{00}} \left(-g_{0\alpha} dx^\alpha + \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta} \right)$$

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}} \sqrt{(g_{0\alpha} g_{0\beta} - g_{\alpha\beta} g_{00}) dx^\alpha dx^\beta}. \tag{III.8}$$

Then

$$dl = \frac{c}{2} d\tau = \frac{c}{2} \frac{\sqrt{g_{00}}}{c} (dx^{0(2)} - dx^{0(1)}) \tag{III.9}$$

and finally

$$dl^2 = \gamma_{\alpha\beta} dx^\alpha dx^\beta, \text{ where } \gamma_{\alpha\beta} = -g_{\alpha\beta} + \frac{g_{0\alpha} g_{0\beta}}{g_{00}}. \tag{III.10}$$

C. Synchronization of clocks

If we want to determine the distance between two not infinitesimally closed points, but points separated by some finite distance we should take an integral $\int dl$ along some path connecting the two points. Obviously, we should take dl over the path at the simultaneous moment of time. Hence, we should first to define what are simultaneous events and then we should synchronize clocks (again with using light signals) over finite volume in space along the path of integration.

The moment at the point B, corresponding to the time coordinate $x^0 + \Delta x^0$, is simultaneous to the moment at the point A, corresponding to the time coordinate x^0, if

$$x^0 + \Delta x^0 = x^0 + \frac{1}{2}(dx^{0(2)} + dx^{0(1)}), \tag{III.11}$$

i.e. the reading of clock in B is halfway between the moments of departure and return of the signal to that point, hence

$$\Delta x^0 = -\frac{g_{0\alpha}}{g_{00}} dx^\alpha. \tag{III.12}$$

As we are able now to define simultaneous events along any open curve, however, synchronization of clocks along a closed contour is impossible in general, since

$$-\oint \frac{g_{0\alpha}}{g_{00}} dx^\alpha \neq 0, \tag{III.13}$$

which means that starting synchronization in some point we return back with

$$\Delta x^0 \neq 0. \tag{III.14}$$

In other words, in an arbitrary reference system the synchronization of clocks in a whole space-time is impossible, but this is not the property of the space-time itself, but the property of the given frame of reference. We always can choose such a frame of reference in which all

$$g_{0\alpha} = 0 \tag{III.15}$$

and hence the synchronization of clocks in a whole space-time is possible. For that we should write 3 equations for 4 arbitrary functions, which is always possible.

D. Invariant 4-volume

To derive EFEs we should be able to calculate integrals over the all space and over the time coordinate

$$S_g = \int G d\tilde{\Omega}, \tag{III.16}$$

where $d\tilde{\Omega}$ is invariant, i.e. not depending on the frame of reference, the element of 4-volume and G is some scalar function. Thus we should understand what the invariant volume is.
Let us prove that the invariant volume is

$$d\tilde{\Omega} = \sqrt{-g} d\Omega, \tag{III.17}$$

where

$$d\Omega = dx^0 dx^1 dx^2 dx^3 \tag{III.18}$$

and g is the determinant of the metric tensor.
Let us first introduce the Jacobian, J, of the transformation from the Galilean (locally inertial) frame of reference, (x'^0, x'^1, x'^2, x'^3), to the curvilinear coordinates (x^0, x^1, x^2, x^3)

$$J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(x'^0, x'^1, x'^2, x'^3)} = |\frac{\partial x^i}{\partial(x'n}| = |S^i_n|, \tag{III.19}$$

where $|A^i_n|$ means the determinant of a matrix A^i_n. Then let us write the formula for the transformation of the contravariant metric tensor

$$g^{ik} = S^i_l S^k_m g^{lm(0)} = S^i_l S^k_m \eta^{lm}, \tag{III.20}$$

where

$$\eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{III.21}$$

Taking into account that the determinant of the reciprocal tensor g^{ik} is the inverse of the determinant of the tensor g_{ik}, we have

$$|g^{ik}| = \frac{1}{|g_{ik}|} = \frac{1}{g}. \tag{III.22}$$

Taking into account that the determinant of the product of matrices is equal to the product of their determinants (the fact known from any textbook on Linear Algebra), we obtain

$$|g^{ik}| = |S^i_l| \times |S^k_m| \times |\eta^{lm}| = J \times J \times (-1) = -J^2, \tag{III.23}$$

hence

$$\frac{1}{g} = -J^2 \text{ and } J = \frac{1}{\sqrt{-g}}. \tag{III.24}$$

From the definition of J we have

$$d\Omega \equiv dx^0 dx^1 dx^2 dx^3 = J dx'^0 dx'^1 dx'^2 dx'^3 = \frac{1}{\sqrt{-g}} dx'^0 dx'^1 dx'^2 dx'^3 = \frac{1}{\sqrt{-g}} d\Omega', \tag{III.25}$$

hence in all curvilinear coordinates

$$\sqrt{-g} d\Omega = d\Omega', \text{ thus } d\tilde{\Omega} = \sqrt{-g} d\Omega \tag{III.26}$$

is invariant 4-volume.

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Lecture 4. *Last updated 07.03.10*

IV. COVARIANT DIFFERENTIATION

A. Parallel translation

In Special Relativity if A_i is a vector dA^i is also a vector (the same is valid for any tensor). But in curvilinear coordinates this is not the case:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k \tag{IV.1}$$

$$dA_i = \frac{\partial x'^k}{\partial x^i} dA'_k + A'_k \frac{\partial^2 x'^k}{\partial x^i \partial x^l} dx^l, \tag{IV.2}$$

thus dA_i is not a vector unless x'^k are linear functions of x^k (like in the case of Lorentz transformations). Let us introduce the following very useful notation:

$$, i = \frac{\partial}{\partial x^i} \tag{IV.3}$$

According to the principle of covariance we can not afford to have not tensors in any physical equations, thus we should replace all differentials like

$$dA_i \text{ and } \frac{\partial A_i}{\partial x^k} \equiv A_{i,k} \tag{IV.4}$$

by some corrected values which we will denote as

$$DA_i \text{ and } A_{i;k} \tag{IV.5}$$

correspondingly. In arbitrary coordinates to obtain a differential of a vector which forms a vector we should subtract vectors in the same point, not in different as we have done before.
Hence, we need produce a parallel transport or a parallel translation. Under a parallel translation of a vector in galilean frame of reference its components don't change, but in curvilinear coordinates they do and we should introduce some corrections:

$$DA^i = dA^i - \delta A^i. \tag{IV.6}$$

These corrections obviously should be linear with respect to all components of A_i and independently they should be linear with respect of dx^k, hence we can write these corrections as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \tag{IV.7}$$

where Γ^i_{kl} are called Christoffel Symbols which obviously don't form any tensor, because DA_i is the tensor while as we know dA_i is not a tensor.

B. Covariant derivatives and Christoffel symbols

In terms of the Christoffel symbols

$$DA^i = (\frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k)dx^l = (A^i_{,l} + \Gamma^i_{kl}A^k)dx^l, \tag{IV.8}$$

$$DA_i = (\frac{\partial A_i}{\partial x^l} - \Gamma^k_{il}A_k)dx^l = (A_{i,l} - \Gamma^k_{il}A_k)dx^l, \tag{IV.9}$$

$$A^i_{;l} = \frac{\partial A^i}{\partial x^l} + \Gamma^i_{kl}A^k = A^i_{,l} + \Gamma^i_{kl}A^k, \tag{IV.10}$$

$$A_{i;l} = \frac{\partial A_i}{\partial x^l} - \Gamma^k_{il}A_k = A_{i,l} - \Gamma^k_{il}A_k. \tag{IV.11}$$

To calculate the covariant derivative of tensor let us start with contravariant tensor which can be presented as a product of two contravariant vectors A^iB^k. In this case the corrections under parallel transport are

$$\delta(A^iB^k) = A^i\delta B^k + B^k\delta A^i = -A^i\Gamma^k_{lm}B^l dx^m - B^k\Gamma^i_{lm}A^l dx^n, \tag{IV.12}$$

since these corrections are linear we have the same for arbitrary tensor A^{ik}:

$$\delta A^{ik} = -(A^{im}\Gamma^k_{ml} + A^{mk}\Gamma^i_{ml})dx^l \tag{IV.13}$$

$$DA^{ik} = dA^{ik} - \delta A^{ik} \equiv A^{ik}_{;\,l}dx^l, \tag{IV.14}$$

hence

$$A^{ik}_{;\,l} = A^{ik}_{,l} + \Gamma^i_{ml}A^{mk} + \Gamma^k_{ml}A^{im} \tag{IV.15}$$

In similar way we can obtain that

$$A^i_{k;\,l} = A^i_{k,l} - \Gamma^m_{kl}A^i_m + \Gamma^i_{ml}A^m_k, \text{ and } A_{ik;\,l} = A_{ik,l} - \Gamma^m_{il}A_{mk} - \Gamma^m_{kl}A_{m,\,i}. \tag{IV.16}$$

In the most general case when we have tensor of $m+n$ rank with m contravariant and n covariant indices the rule for calculation of the covariant derivative with respect to index p is the following

$$A^{i_1\,i_2\,\dots\,i_m}_{j_1\,j_2\,\dots\,j_n\,;\,\mathbf{p}} = A^{i_1\,i_2\,\dots\,i_m}_{j_1\,j_2\,\dots\,j_n\,,\,\mathbf{p}} + \Gamma^{\mathbf{i_1}}_{\mathbf{kp}}A^{\mathbf{k}\,i_2\,\dots\,i_m}_{j_1\,j_2\,\dots\,j_n} + \Gamma^{\mathbf{i_2}}_{\mathbf{kp}}A^{i_1\,\mathbf{k}\,\dots\,i_m}_{j_1\,j_2\,\dots\,j_n} + \dots + \Gamma^{\mathbf{i}_m}_{\mathbf{kp}}A^{i_1\,i_2\,\dots\,\mathbf{k}}_{j_1\,j_2\,\dots\,j_n} - \tag{IV.17}$$

$$-\Gamma^{\mathbf{k}}_{\mathbf{j_1}\,\mathbf{p}}A^{i_1\,i_2\,\dots\,i_m}_{\mathbf{k}\,j_2\,\dots\,j_n} - \Gamma^{\mathbf{k}}_{\mathbf{j_2}\,\mathbf{p}}A^{i_1\,i_2\,\dots\,i_m}_{j_1\,\mathbf{k}\,\dots\,j_n} - \dots - \Gamma^{\mathbf{k}}_{\mathbf{j}_n\,\mathbf{p}}A^{i_1\,i_2\,\dots\,i_m}_{j_1\,j_2\,\dots\,\mathbf{k}}. \tag{IV.18}$$

C. The Christoffel symbols and the metric tensor

So far we don't know how the Christoffel symbols depend on coordinates, however we can prove that they are symmetric in the subscripts. Let some covariant vector A_i is the gradient of a scalar ϕ, i.e. $A_i = \phi_{,i}$. Then

$$A_{k;\,i} - A_{i;\,k} = \phi_{,k,i} - \Gamma^l_{ki}\phi_{,l} - \phi_{,i,k} + \Gamma^l_{ik}\phi_{,l} = \left(\Gamma^l_{ki} - \Gamma^l_{ik}\right)\phi_{,l}. \tag{IV.19}$$

In Galilean coordinates

$$\Gamma^l_{ik} = \Gamma^l_{ki} = 0, \quad \text{hence in Galilean coordinates } A_{k;\,i} - A_{i;\,k} = 0, \tag{IV.20}$$

but taking into account that $A_{k;\,i} - A_{i;\,k}$ is a tensor we conclude that if it equals to zero in one system of coordinates it should be equal to zero in any other coordinate system, hence

$$\Gamma^l_{ik} = \Gamma^l_{ki} \tag{IV.21}$$

in any coordinate system.
This is a typical example of the proof widely used in General Relativity:
If some equality between tensors is valid in one coordinate system then this equality is valid in arbitrary coordinate system. This is obvious advantage to deal with tensors.
Then we can show that covariant derivatives of g_{ik} are equal to zero. Indeed:

$$DA_i = g_{ik}DA^k \quad DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^k Dg_{ik}, \quad \text{hence} \quad g_{ik}DA^k = g_{ik}DA^k + A^k Dg_{ik}, \tag{IV.22}$$

which obviously means that

$$A^k Dg_{ik} = 0. \tag{IV.23}$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0. \tag{IV.24}$$

This is another example of proof in General Relativity: If the the sum $B_{ik}A^i = 0$ for arbitrary vector A^i then the tensor $B_{ik} = 0$. Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0 \tag{IV.25}$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0. \tag{IV.26}$$

Now we are ready to relate the Christoffel symbols to the metric tensor. Introducing useful notation

$$\Gamma_{k,\,il} = g_{km}\Gamma^m_{il}, \tag{IV.27}$$

we have

$$g_{ik;\,l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\,il} - \Gamma_{i,\,kl} = 0. \tag{IV.28}$$

Permuting the indices i, k and l twice as $i \to k$, $k \to l$, $l \to i$,we obtain

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\,il} + \Gamma_{i,\,kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\,kl} + \Gamma_{l,\,ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\,ki} - \Gamma_{k,\,li}. \tag{IV.29}$$

Taking into account that $\Gamma_{k,\,il} = \Gamma_{k,\,li}$, after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl}, \tag{IV.30}$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m}\right). \tag{IV.31}$$

Now we have expressions for the Christoffel symbols in terms of the metric tensor and hence we know their dependence on coordinates.

D. Physical applications

The previous material can be summarized as follows:
Gravity is equivalent to curved space-time, hence in all differentials of tensors we should take into account the change in the components of a tensor under an infinitesimal parallel transport. Corresponding corrections are expressed in terms of the Cristoffel symbols and are reduced to replacement of any partial derivative by corresponding covariant derivative. In other words we can say that if one wants to take into account all effects of Gravity on any local physical process, described by the corresponding equations written in framework of Special Relativity, one should just replace all partial derivatives by covariant derivatives in these equation according to the following very nice and simple but actually very strong and important formulae:

$$\mathbf{d} \to \mathbf{D} \quad \text{and} \quad , \to ;. \tag{IV.32}$$

Let us consider only a few examples.

1. Application of (IV.32) to the metric tensor itself

In special Relativity

$$dg_{ik} = 0 \ \text{ and } \ g_{ik,l} = 0, \tag{IV.33}$$

while in General Relativity

$$Dg_{ik} = 0 \ \text{ and } \ g_{ik;l} = 0. \tag{IV.34}$$

2. Application of (IV.32) to the motion of test particle

Let us apply above formulae to Let

$$u^i = \frac{dx^i}{ds} \tag{IV.35}$$

is the four-velocity. Then the equation for motion of a free particle in absence of gravitational field is

$$\frac{du^i}{ds} = 0 \tag{IV.36}$$

is generalized to the equation

$$\frac{Du^i}{ds} = 0, \tag{IV.37}$$

which gives

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{kn} u^k \frac{dx^n}{ds} = \frac{d^2x^i}{ds^2} + \Gamma^i_{kn} u^k u^n = 0. \tag{IV.38}$$

Thus from physical point of view the equation

$$\frac{d^2x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0 \tag{IV.39}$$

describes the motion of free particle in a given gravitational field and

$$\frac{d^2x^i}{ds^2} = -\Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} \tag{IV.40}$$

is the four-acceleration, while from geometrical point of view this equation is the equation for geodesics in a curved space-time. That is why all particles move with the same acceleration and now this experimental fact is not coincidence anymore but consequence of geometrical interpretation of gravity.

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Lecture 5. *Last updated 07.03.10*

V. MOTION OF A TEST PARTICLE IN A GRAVITATIONAL FIELD

A. Hamilton-Jacobi equation

Any object of a small enough mass is called a test particle. Small mass means that gravitational field generated by this object is negligible in comparison with the external gravitational field generated by other, much more massive, objects. The role of such test particle can be played by a planet around a star or a star around a massive black hole, or by photon propagating around a neutron star or black hole.
From the previous lecture we know that the motion of particles and photons in a given gravitational field is described by the space-time geodesics. The geodesic equations are very useful for physical understanding of the motion of particles and propagation of photons; however, it is easier to work with the Hamilton−Jacobi equation. The advantage of this approach is that it equates the motion of particles with the propagation of waves.
The derivation of Hamilton−Jacobi equation is really very simple. From the definition of the four-velocity

$$u^i = \frac{dx^i}{ds}, \tag{V.1}$$

we have

$$ds^2 = g_{ik}dx^i dx^k = g_{ik}u^i u^k ds^2 = u_i u^i ds^2, \tag{V.2}$$

hence

$$u^i u_i = 1. \tag{V.3}$$

Four-momentum of the particle is defined as

$$p^i = mcu^i, \ \text{ hence } \ p_i p^i = g^{ik}p_i p_k = m^2c^2. \tag{V.4}$$

Taking into account that a covariant vector transforms as the gradient of a scalar, we can introduce such a scalar function that

$$p_i = -\frac{\partial S}{\partial x^i}, \tag{V.5}$$

then we immediately obtain the Hamilton−Jacobi Equation for a particle in a gravitational field

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m^2c^2 = 0. \tag{V.6}$$

B. Eikonal equation

The equation for the geodesic obtained in Lecture 4 is not applicable to the propagation of light since $ds = 0$. However, we can introduce some scalar parameterλ varying along world line of the light signal and then introduce a vector

$$k^i = \frac{dx^i}{d\lambda}, \tag{V.7}$$

which is tangent to the word line. This vector is called the four- dimensional wave vector. In the absence of a gravitational field according to the geometrical optics the propagation of light is given by the equation

$$dk^i = 0. \tag{V.8}$$

We know that the generalization of this equation in General Relativity is straightforward: $d \to D$. Then from $Dk^i = 0$ we obtain

$$\frac{dk^i}{d\lambda} + \Gamma^i_{kl} k^k k^l = 0. \tag{V.9}$$

From the definition of the four-vector for light (V.7) we have

$$ds^2 = g_{ik} dx^i dx^k = g_{ik} k^i k^k d\lambda^2, \tag{V.10}$$

then taking into account that $ds = 0$, we obtain

$$k_i k^i = g^{ik} k_i k_k = 0. \tag{V.11}$$

We know that any covariant vector can be presented as the gradient of a scalar

$$k_i = -\frac{\partial \Psi}{\partial x^i}, \tag{V.12}$$

were Ψ is a scalar. And we immediately obtain the Eikonal Equation in gravitational field

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0. \tag{V.13}$$

The physical meaning of Ψ, which is called the Eikonal, follows from the obvious relationship

$$\Psi = -\int k_i dx^i, \tag{V.14}$$

which looks like the phase of the electromagnetic wave. We can see that the General Relativity can easily solve the problem of propagation of electromagnetic signals in the presence of a gravitational field, while the Newtonian gravity can not even offer more or less self consistent approach to the problem.
The shortest way to obtain the Eikonal equation is just to put $m = 0$ in the HamiltonJacobi equation and change notations.

C. The motion in a spherically symmetric static gravitational field

As an example of the motion of a test particle in a given gravitational field, let us consider a spherically symmetric gravitational field and assume that this field does not depend on time, i.e. it is static field. Taking into account the spherical symmetry we can choose our spherical coordinates in a such way that the plane of orbit coincides with the equatorial plane $\theta = \pi/2$ and $d\theta = 0$. Obviously, all the components of a metric tensor are functions of the radial coordinate only. Let us denote the radial coordinate as $x^1 = r$.
We can write the interval describing such gravitational field as

$$ds^2 = g_{00}(r) c^2 dt^2 + g_{11}(r) dr^2 + g_{33} d\phi^2. \tag{V.15}$$

In this case the Hamilton−Jacobi equation can be written as

$$g^{00}(r)\left(\frac{\partial S}{c\partial t}\right)^2+g^{11}(r)\left(\frac{\partial S}{\partial r}\right)^2+g^{33}(r)\left(\frac{\partial S}{\partial \phi}\right)^2-m^2c^2=0. \tag{V.16}$$

Since all coefficients in this equation do not depend on t and ϕ we can say that

$$\frac{\partial S}{\partial t}=-E, \text{ and } \frac{\partial S}{\partial \phi}=L, \tag{V.17}$$

where E and L are constants, which by definition are the energy and angular momentum of the particle under consideration. Then putting

$$S=-Et+L\phi+S_r(r) \tag{V.18}$$

into the Hamilton−Jacobi equation we have

$$g^{00}(r)\frac{E^2}{c^2}+g^{11}(r)\left(\frac{dS_r(r)}{dr}\right)^2+g^{33}(r)L^2-m^2c^2=0, \tag{V.19}$$

hence

$$g^{11}(r)\left(\frac{dS_r(r)}{dr}\right)^2=-g^{00}(r)\frac{E^2}{c^2}-g^{33}(r)L^2+m^2c^2, \tag{V.20}$$

and

$$\frac{dS_r(r)}{dr}=\pm\sqrt{-\frac{1}{g^{11}(r)}\left(g^{00}(r)\frac{E^2}{c^2}+g^{33}(r)L^2-m^2c^2\right)}=\pm mc\sqrt{-\frac{g^{00}(r)}{g^{11}(r)}\left(\tilde{E}^2+\frac{g^{33}(r)}{g^{00}(r)}\tilde{L}^2-\frac{1}{g^{00}(r)}\right)}, \tag{V.21}$$

where

$$\tilde{E}=\frac{E}{mc^2} \text{ and } \tilde{L}=\frac{L}{mc}. \tag{V.22}$$

Then we can calculate the radial component of the 4-velocity:

$$u^1\equiv\frac{dr}{ds}=\frac{p^1}{mc}=g^{11}(r)\frac{p_1}{mc}=-g^{11}(r)\frac{\partial S}{mc\partial r}=-g^{11}(r)\frac{dS_r(r)}{mcdr}=\mp\sqrt{-g^{00}(r)g^{11}(r)\left(\tilde{E}^2-U^2(r)_{eff}\right)}, \tag{V.23}$$

where

$$U^2_{eff}(r)=\frac{1}{g^{00}(r)}\left(1-g^{33}(r)\tilde{L}^2\right) \tag{V.24}$$

is the so called "effective" potential. One can see that the condition

$$\frac{E}{mc^2}>U_{eff} \tag{V.25}$$

determines the admissible range of the motion. The effective potential includes in the relativistic manner potential energy plus kinetic energy of non-radial motion, this kinetic energy is determined by angular momentum L.
The radius of stable and unstable circular orbits is obtained from the simultaneous solution of the equations

$$U_{eff}=\tilde{E} \;\; and \;\; \frac{dU_{eff}}{dr}=0. \tag{V.26}$$

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Lecture 6. *Last updated 07.03.10*

VI. CURVATURE OF SPACE-TIME

A. The Riemann curvature tensor

We know that $A_{i,k,l} - A_{i,l,k} = 0$. What can we say about the following commutator $A_{i;\,k;\,l} - A_{i;\,l;\,k}$? Straightforward calculations will show that this is not equal to zero in the presence of gravitational field and can be presented as

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}, \tag{VI.1}$$

where the object R^i_{klm} is obviously a tensor and called the curvature Riemann tensor.
We know that if at least one component of a tensor is not equal to zero in at least one frame of reference, the same is true for any other frame of reference. In other words, tensors (in contrast to the Christoffel symbols) can not be eliminated by transformations of coordinates.
The Riemann tensor describes an actual tidal gravitational field, which is not local and, hence, can not be eliminated even in the locally inertial frame of reference. Let us calculate the curvature Riemann tensor directly:

$$A_{i;k;l} - A_{i;l;k} = A_{i;k,l} - A_{i;l,k} - \Gamma^m_{il} A_{m;k} - \Gamma^m_{kl} A_{i;m} + \Gamma^m_{ik} A_{m;l} + \Gamma^m_{lk} A_{i;m} =$$

$$= \left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} - \Gamma^m_{il}\left(A_{m,k} - \Gamma^p_{mk} A_p\right) + \Gamma^m_{ik}\left(A_{m,l} - \Gamma^p_{ml} A_p\right) =$$

$$= A_{i,k,l} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{ik,l} A_m - A_{i,l,k} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{il,k} A_m - \Gamma^m_{il} A_{m,k} + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{ik} A_{m,l} - \Gamma^m_{ik}\Gamma^p_{ml} A_p =$$

$$= -\Gamma^m_{ik,l} A_m + \Gamma^m_{il,k} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p - \Gamma^m_{ik}\Gamma^p_{ml} A_p = -\Gamma^m_{ik,l} A_m + \Gamma^m_{il,k} A_m + \Gamma^p_{il}\Gamma^m_{pk} A_m - \Gamma^p_{ik}\Gamma^m_{pl} A_m =$$

$$= \left(-\Gamma^m_{ik,l} + \Gamma^m_{il,k} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl}\right) A_m = R^m_{ikl} A_m. \tag{VI.2}$$

Finally

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl}. \tag{VI.3}$$

Similar equations can be written for tensors of higher ranks, for example

$$A_{ik;\,l;\,m} - A_{ik;\,m;\,l} = A_{in} R^n_{klm} + A_{nk} R^n_{ilm}. \tag{VI.4}$$

Let us introduce the covariant presentation of the Riemann tensor:

$$R_{iklm} = g_{in} R^n_{klm}. \tag{VI.5}$$

By straightforward calculations one can show that

$$R_{iklm} = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) + g_{np}(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}). \tag{VI.6}$$

B. Symmetry properties of the Riemann tensor

There are several symmetry properties of the curvature tensor:
1) The Riemann tensor is antisymmetric with respect to permutations of indices within each pair

$$R_{iklm} = -R_{kilm} = -R_{ikml}. \tag{VI.7}$$

2) The Riemann tensor is symmetric with respect to permutations of pairs of indices

$$R_{iklm} = R_{lmik}. \tag{VI.8}$$

3) The cyclic sum formed by permutation of any three indices is equal to zero

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0. \tag{VI.9}$$

C. Bianchi Identity

The most important property of the Riemann tensor is so called the Bianchi identity:

$$R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = 0. \tag{VI.10}$$

It is easy to verify this identity in a locally inertial frame of reference, where

$$\Gamma^i_{kl} = 0, \tag{VI.11}$$

hence

$$R^n_{ikl;\,m} + R^n_{imk;\,l} + R^n_{ilm;\,k} = R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = \tag{VI.12}$$

$$\Gamma^n_{il,m,k} - \Gamma^n_{ik,m,l} + \Gamma^n_{ik,l,m} - \Gamma^n_{im,l,k} + \Gamma^n_{im,k,l} - \Gamma^n_{il,k,m} = 0. \tag{VI.13}$$

Taking into account that the Bianchi identity is of a tensor character, we can conclude that it valid in any other frame of reference.

D. The Ricci tensor and the scalar curvature

Now we can introduce a second rank curvature tensor, called the Ricci tensor, as follows

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk}. \tag{VI.14}$$

We can also introduce a zero rank curvature tensor, i.e. a scalar, called the scalar curvature:

$$R = g^{ik} R_{ik}. \tag{VI.15}$$

1. *The important consequence of Bianchi identity*

After contracting the Biancci identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0 \tag{VI.16}$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0. \tag{VI.17}$$

According to the definition of Ricci tensor (VI.14), the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}. \tag{VI.18}$$

Taking into account that the Riemann tensor is antisymmetric with respect to permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km}, \tag{VI.19}$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}. \tag{VI.20}$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip} R_{pklm;i}, \tag{VI.21}$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip} R_{pklm;i} = -g^{ip} R_{kplm;i} = g^{ip} R_{kpml;i}. \tag{VI.22}$$

After all these manipulations we have

$$g^{ip} R_{kpml;i} + R_{kl;m} - R_{km;l} = 0. \tag{VI.23}$$

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km} g^{ip} R_{kpml;i} + g^{km} R_{kl;m} - g^{km} R_{km;l} = \left(g^{km} g^{ip} R_{kpml}\right)_{;i} + \left(g^{km} R_{kl}\right)_{;m} - \left(g^{km} R_{km}\right)_{;l} = 0. \tag{VI.24}$$

In the first term expression in brackets can be simplified as

$$g^{km} g^{ip} R_{kpml} = g^{ip} R_{pl} = R^i_l. \tag{VI.25}$$

In the second term the expression in brackets can be simplified as

$$g^{km} R_{kl} = R^m_l. \tag{VI.26}$$

According to the definition of the scalar curvature (VI.15), the third term can be simplified as

$$\left(g^{km} R_{km}\right)_{;l} = R_{;l} = R_{,l}. \tag{VI.27}$$

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0, \tag{VI.28}$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \quad \text{or} \quad R^i_{l;i} - \frac{1}{2} R_{,l} = 0. \tag{VI.29}$$

Thus the important consequence of Bianchi identity is

$$R^i_{l;i} - \frac{1}{2} R_{,l} = 0. \tag{VI.30}$$

E. Geodesic deviation equation

The geodesic deviation equation is an equation involving the Riemann curvature tensor, which measures the change in separation of neighboring geodesics. In the language of mechanics it measures the rate of relative acceleration of two particles moving forward on neighboring geodesics. Let the 4-velocity along one geodesic is

$$u^i = \frac{dx^i}{ds}. \tag{VI.31}$$

There is an infinitesimal separation vector between the two geodesics η^i. Then the relative acceleration, a^i, is

$$a^i = \frac{d^2\eta^i}{ds^2}. \tag{VI.32}$$

It is possible to show that

$$a^i = R^i_{klm} u^k u^l \eta^m. \tag{VI.33}$$

If gravitational field is weak and all motions are slow

$$u^i \approx \delta^i_0, \tag{VI.34}$$

and the above equation is reduced to the Newtonian equation for the tidal acceleration.

F. Stress-Energy Tensor

The stress-energy tensor (sometimes stress-energy-momentum tensor), T_{ik}, describes the density and flux of energy and momentum.
In general relativity this tensor is symmetric and contains ten independent components:
The component T_{00} represents the energy density (1 component).
The components $T_{0\alpha}$ ($\alpha = 1, 2, 3$) represent the flux of energy across the surface which is normal to the x_α-axis. These components are equivalent to the components $T_{\alpha 0}$ which describe the density of the α^{th} momentum (3 components).
The components $T_{\alpha\beta}$ (α, $\beta = 1, 2, 3$) represent flux of α^{th} momentum across the surface which is normal to the x_β-axis.
In particular, the diagonal components $T_{\alpha alpha}$ represents a pressure-like quantity, normal stress (3 components).
Non-diagonal components $T_{\alpha\beta}$ ($\alpha \neq \beta$), represent shear stress (3 components).
All these ten components participate in the generation of a gravitational field, while in Newton gravity the only source of gravitational field is the mass density.

1. *Conservation of energy-momentum in gravitational field*

According to physics in absence of gravitational field the stress-energy tensor satisfies the following conservation law:

$$T^i_{k,i} = 0. \tag{VI.35}$$

We know from previous lectures that according ", $\to$;-rule" in the presence of gravitational field this should be rewritten as

$$T^i_{k;i} = 0. \tag{VI.36}$$

G. Heuristic "Derivation" of EFEs

It seems like a good idea to relate the Ricci tensorto the stress-energy tensor.
The most general form of the second rank tensor formed from the metric tensor g_{ik} and containing second derivatives of the metric tensor g_{ik}, let us call it the Einstein tensor, is

$$G_{ik} = R_{ik} + \alpha g_{ik} R. \tag{VI.37}$$

As follows from the the previous section

$$G^i_{k;i} = (g^{in} G_{nk})_{;i} = R^i_{k;i} + \alpha \delta^i_k R_{,i} = (\frac{1}{2} + \alpha) R_{,k}. \tag{VI.38}$$

Let us assume that the EFEs have the following form

$$G_{ik} = \kappa T_{ik}, \tag{VI.39}$$

where the constant κ is called the Einstein constant. Multiplying this by g^{mk} we obtain

$$R^m_i + \alpha \delta^m_i R = \kappa T^m_k. \tag{VI.40}$$

Taking covariant divergence of LHS and RHS of this equation we obtain

$$(\alpha + \frac{1}{2}) R_{;k} = \kappa T^m_{k;\,m} = 0, \tag{VI.41}$$

hence

$$\alpha = -\frac{1}{2}, \tag{VI.42}$$

and final EFEs are

$$R^i_k - \frac{1}{2} \delta^i_k R = \kappa T^i_k. \tag{VI.43}$$

To determine κ we can use the so called the correspondence principle, which says that the EFEs in weak-field and the slow-motion approximation should be reduced to Newton's law of gravity, i.e. to the Poisson's equation

$$\Delta \phi = 4\pi G \rho. \tag{VI.44}$$

By straightforward calculations one can prove that such reduction is possible only if

$$\kappa = \frac{8\pi G}{c^4}. \tag{VI.45}$$

Finally, EFEs can be written as

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik}. \tag{VI.46}$$

Despite the simple appearance of this equation it is, in fact, quite complicated. Given a specified distribution of matter and energy in the form of a stress-energy tensor, the EFE are understood to be equations for the metric tensor g_{ik}, as both the Ricci tensor and Ricci scalar depend on the metric (in a complicated nonlinear manner). In fact, when fully written out, the EFEs are the system of 10 coupled, nonlinear, hyperbolic-elliptic partial differential equations. In other words, Despite the simple appearance of the EFEs they are, in fact, rather complicated.
Solutions of the Einstein field equations model an extremely wide variety of gravitational fields.
Some of them are really exotic, for example the solution corresponding to the so called **wormhole** [A wormhole is a hypothetical topological feature of spacetime that is essentially a 'shortcut' through space and time. A wormhole has at least two mouths which are connected to a single throat. If the wormhole is traversable, matter can 'travel' from one mouth to the other by passing through the throat. While there is no observational evidence for wormholes, spacetimes containing wormholes are known to be valid solutions of the Einsteins equations.
Gravitational waves and **black holes** are also solutions of EFEs.

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Lecture 7. *Last updated 07.03.10*

VII. RIGOROUS DERIVATION OF EFES

A. The principle of the least action

The derivation of EFEs is very important material for understanding GR. In this lecture we will derive rigorously the Einstein Field equations (EFEs) from the principle of the least action.
This principle says that

$$\delta(S_g + S_m) = 0, \tag{VII.1}$$

where S_g and S_m are the actions of gravitational field and matter respectively. Taking into account that we are going to derive EFEs, the subject of variations is all components of the metric tensor.
To derive EFEs we should understand what are S_g and S_m .

B. The action function for the gravitational field

First of all S_g should depend on configuration of gravitational field, or geometry, in the whole space-time, hence it should be expressed in terms of a scalar integral over the all space and over the time coordinate between two given moments of time

$$S_g = \int G d\tilde{\Omega}, \tag{VII.2}$$

where $d\tilde{\Omega}$ is invariant element of 4-volume (see Lecture 3) and G is some scalar function called the action density.
We know that the final equations should contain derivatives of g_{ik} no higher than the second. Otherwise we could not obtain Newtonian Poisson's equation (see the previous lecture). In other words, G must contain only g_{ik} and Γ^l_{mn}, i.e

$$G = G(g_{ik}, \Gamma^i_{kl}). \tag{VII.3}$$

Immediately we confront with the following problem : this is impossible to construct the scalar from g_{ik} and Γ^l_{mn} . The only scalar in gravitational field, the scalar curvature R, contains the second derivatives of g_{ik}. Fortunately, there is rather simple resolution of this paradox: R is linear with respect to the second derivatives and for this reason, as we will see later, all terms containing second derivatives don't contribute to the variations of the action. Let us write the action function in the following form

$$S_g = \alpha \int R\sqrt{-g} d\Omega, \tag{VII.4}$$

where α is a constant which will be determined later.

Because of the linearity of R with respect to the second derivatives, the invariant action function can be transformed in the following way

$$S_g = \alpha \int R\sqrt{-g}d\Omega = \alpha \int G\sqrt{-g}d\Omega + \alpha \int w^l_{,l} d\Omega, \tag{VII.5}$$

where G contains only g_{ik} and $g_{ik,n}$, w is a function which we can be obtained by straightforward calculations:

$$\sqrt{-g}R = \sqrt{-g}g^{ik}R_{ik} = \sqrt{-g}\left\{g^{ik}\Gamma^l_{ik,l} - g^{ik}\Gamma^l_{il,k} + g^{ik}\Gamma^l_{ik}\Gamma^m_{lm} - g^{ik}\Gamma^m_{il}\Gamma^l_{km}\right\}, \tag{VII.6}$$

obviously

$$\sqrt{-g}g^{ik}\Gamma^l_{ik,l} = (\sqrt{-g}g^{ik}\Gamma^l_{ik})_{,l} - \Gamma^l_{ik}(\sqrt{-g}g^{ik})_{,l} \tag{VII.7}$$

and

$$\sqrt{-g}g^{ik}\Gamma^l_{il,k} = (\sqrt{-g}g^{ik}\Gamma^l_{il})_{,k} - \Gamma^l_{il}(\sqrt{-g}g^{ik})_{,k} = (\sqrt{-g}g^{il}\Gamma^k_{ik})_{,l} - \Gamma^k_{ik}(\sqrt{-g}g^{il})_{,l}. \tag{VII.8}$$

Then we obtain

$$\sqrt{-g}R = (\sqrt{-g}g^{ik}\Gamma^l_{ik} - \sqrt{-g}g^{il}\Gamma^k_{ik})_{,l} + \sqrt{-g}G = w^l, l + \sqrt{-g}G, \tag{VII.9}$$

where

$$w^l = \sqrt{-g}(g^{ik}\Gamma^l_{ik} - g^{il}\Gamma^k_{ik}) \tag{VII.10}$$

and

$$\sqrt{-g}G = \Gamma^m_{im}(\sqrt{-g}g^{ik}), k - \Gamma^l_{ik}(\sqrt{-g}g^{ik})_{,l} - (\Gamma^m_{il}\Gamma^l_{km} - \Gamma^l_{ik}\Gamma^m_{lm})\sqrt{-g}g^{ik} \tag{VII.11}$$

$$\Gamma^i_{ki} = \frac{1}{2}g^{im}\frac{\partial g_{im}}{\partial x^k}. \tag{VII.12}$$

According to the Gauss' theorem the volume integral of a full derivative is reduced to the integral over boundary. Taking into account that our objective is to obtain proper equations by applying the principle of the least action, we should keep all boundary conditions fixed. Hence, w disappears after variation. As a result

$$\delta \int R\sqrt{-g}d\Omega = \delta \int G\sqrt{-g}d\Omega. \tag{VII.13}$$

Thus we don't need G any more, because we proved that the variation of the integral with R is the same as the variation of the integral with G, hence we can work with R only.

$$\delta \int R\sqrt{-g}d\Omega = \delta \int g^{ik}R_{ik}\sqrt{-g}d\Omega = \int \{R_{ik}\sqrt{-g}\delta g^{ik} + g^{ik}R_{ik}\delta(\sqrt{-g}) + g^{ik}\sqrt{-g}\delta R_{ik}\}d\Omega. \tag{VII.14}$$

There are three terms in the variation of the action function. Let us first calculate the second term.

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}}\delta g = -\frac{1}{2\sqrt{-g}}\frac{\partial g}{\partial g_{ik}}\delta g_{ik} = -\frac{1}{2\sqrt{-g}}M^{ik}\delta g_{ik}, \tag{VII.15}$$

where M^{ik} is the minor of the determinant g corresponding to the component g_{ik}. Indeed, the determinant g depends on all components g_{ik}. Calculating g with the help, say the first raw, one can write $g = M^{1i}g_{1i}$, where M^{1i} are minors of the components in the first row. Obviously M^{1i} do not contain g_{1i}. Hence

$$\frac{\partial g}{\partial g_{1i}} = M^{1i}. \tag{VII.16}$$

This is true for any row in determinant:

$$\frac{\partial g}{\partial g_{ni}} = M^{ni}. \tag{VII.17}$$

Taking into account that g^{ik} is reciprocal to g_{ik}, i.e. $g_{ik}g^{kn} = \delta_i^n$, ($g^{ik}$ is inverse matrix of g_{ik}), one can write $g^{ik} = M^{ik}/g$, i.e. $M^{ik} = gg^{ik}$. Thus

$$dg = \frac{\partial g}{\partial g_{ik}} dg_{ik} = M^{ik} dg_{ik} = gg^{ik} dg_{ik}, \tag{VII.18}$$

hence

$$g^{ik} dg_{ik} = \frac{dg}{g} = d\ln|g| = d\ln(-g) = 2\ln\sqrt{-g}. \tag{VII.19}$$

Then $g^{ik}dg_{ik} = d(g^{ik}g_{ik}) - g_{ik}dg^{ik} = d\delta_i^i - g_{ik}dg^{ik} = -g_{ik}dg^{ik}$.
Thus

$$\delta(\sqrt{-g}) = -\frac{1}{2\sqrt{-g}} gg^{ik}\delta g_{ik} = \frac{1}{2\sqrt{-g}} gg_{ik}\delta g^{ik} = -\frac{1}{2}\sqrt{-g} g_{ik}\delta g^{ik}. \tag{VII.20}$$

Now we can rewrite the variation of action as

$$\delta \int R\sqrt{-g} d\Omega = \int [(R_{ik} - \frac{1}{2} g_{ik}R)\sqrt{-g}\delta g^{ik} + g^{ik}\sqrt{-g}\delta R_{ik}] d\Omega. \tag{VII.21}$$

Let us consider now the last term in the variation. For the calculation of δR_{ik} we can use the fact that although Γ^i_{kn} is not a tensor, its variation, $\delta\Gamma^i_{kn}$, is a tensor.
Proof: Let A^i is an arbitrary vector at the point x^i. After the parallel transport From the point x^i to the point $x^i + dx^i$, as we know, its components are

$$A^i(x^n + dx^n) = A^i(x^n) + (A^i_{,m}(x^n) + \Gamma^i_{mp}(x^n)A^p(x^n))dx^m. \tag{VII.22}$$

Then

$$\delta A^i(x^n + dx^n) = \delta\Gamma^i_{mp}(x^n)A^p(x^n))dx^m. \tag{VII.23}$$

The left side is a vector because it is the difference between two vectors in the same point, hence the right side is also a vector. Thus $\delta\Gamma^i_{mp}(x^n)$ is a tensor.
In a locally galilean frame of reference

$$g^{ik}\delta R_{ik} = g^{ik}\left\{\delta\Gamma^l_{ik,l} - \delta\Gamma^l_{il,k}\right\} = g^{ik}\delta\Gamma^l_{ik,l} - g^{il}\delta\Gamma^k_{ik,l} = W^l_{,l}, \tag{VII.24}$$

where

$$W^l = g^{ik}\delta\Gamma^l_{ik} - g^{il}\delta\Gamma^k_{ik}, \tag{VII.25}$$

obviously W^l is a vector.
Now let us prove that the covariant divergence of an arbitrary vector can be written as follows

$$A^n_{;\,n} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^n)_{,n}. \tag{VII.26}$$

Proof:

$$A^n_{;n} = A^n_{,n} + \Gamma^n_{ni}A^i = A^n_{,n} + \frac{1}{2}g^{nm}(g_{nm,i} + g_{mi,n} - g_{in,m})A^i = A^n_{,n} + \frac{1}{2}(g^{nm}g_{nm,i} + g^{nm}g_{mi,n} - g^{nm}g_{ni,m})A^i =$$

$$= A^n_{,n} + \frac{1}{2}g^{nm}g_{nm,i}A^i. \tag{VII.27}$$

Taking into account (VII.18), one obtains

$$A^i_{;i} = A^n_{,n} + \frac{g_{,n}}{2g}A^n = \frac{1}{\sqrt{-g}}[\sqrt{-g}A^n_{,n} + (\sqrt{-g})_{,n}A^n] = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}. \tag{VII.28}$$

As follows from the proof above, in local galilean frame of reference, where $g = -1$

$$A^i_{;i} = A^i_{,i}, \tag{VII.29}$$

hence, returning back to δR_{ik}, in local galilean frame of reference we have

$$g^{ik}\delta R_{ik} = W^l_{,l} = W^l_{;l}. \tag{VII.30}$$

Since this is a relation between two tensors (of 0-rank), once this is valid in one frame of reference it is valid in an arbitrary frame of reference. Hence

$$\sqrt{-g}g^{ik}\delta R_{ik} = \sqrt{-g}W^l_{;l} = (\sqrt{-g}W^l)_{,l}, \tag{VII.31}$$

this means that according to the Gauss theorem the contribution of the third term in the variation of the action function is equal to zero.
Finally we obtain

$$\delta S_g = \alpha \int (R_{ik} - \frac{1}{2}g_{ik}R)\delta g^{ik}\sqrt{-g}d\Omega. \tag{VII.32}$$

C. The action function for matter

Similar to the action function for gravitational field, the action function for matter can be written as

$$S_m = \int \Psi\sqrt{-g}d\Omega, \tag{VII.33}$$

where Ψ is a scalar action density (by matter we mean any substance including all physical fields, for example, electromagnetic field).
Let us calculate the variation of S_m. Immediately the following problem arises. Obviously Ψ can depend on many physical parameters describing the physical system to which we are trying to apply the least action method. let us denote all of them as q_a, $a = 1, 2, 3, 4, ...$ Should we take into account the variations of all these q_a? The answer is no, all these variations should cancel each other by virtue of the "equations of motion" of the physical system under consideration, since these equations are obtained, according to the principle of the least action, from the condition that the variations of S_m, related with the variations of q_a, are equal to zero. Thus it is enough to take into account the variations of the metric tensor only. Then we have

$$\delta S_m = \int \left\{ \frac{\partial\sqrt{-g}\Psi}{\partial g^{ik}}\delta g^{ik} + \frac{\partial\sqrt{-g}\Psi}{\partial(g^{ik}_{,l})}\delta(g^{ik}_{,l}) \right\} d\Omega. \tag{VII.34}$$

Then taking into account that

$$\delta(g^{ik}_{,l}) = (\delta g^{ik})_{,l}, \tag{VII.35}$$

which means that the partial differentiation, obviously, commutates with the procedure of taking variations, we can integrate the second term in the previous formula by parts, as a result we obtain

$$\delta S_m = \int \left\{ \frac{\partial\sqrt{-g}\Psi}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\Psi}{\partial(g^{ik}_{,l})} \right\} \delta g^{ik}d\Omega. \tag{VII.36}$$

Let us introduce the following notation

$$\sqrt{-g}\ A_{ik} = \frac{\partial\sqrt{-g}\Psi}{\partial g^{ik}} - \frac{\partial}{\partial x^l}\frac{\partial\sqrt{-g}\Psi}{\partial(g^{ik}_{,l})}. \tag{VII.37}$$

Then δS_m takes the following form

$$\delta S_m = \int A_{ik}\delta g^{ik}\sqrt{-g}d\Omega. \tag{VII.38}$$

D. The stress-energy tensor and the action density

One can prove that the tensor A_{ik} introduced in the previous section, is proportional to the stress-energy tensor T_{ik} introduced in the previous lecture.
Proof: Let us carry out infinitesimally small translation from the coordinates x^i to the coordinates $x'^i = x^i + \xi^i$, where ξ^i are infinitesimally small quantities. Considering this translation as a transformation of coordinates, we can see that the contravariant metric tensor is transformed under these translations as

$$g'^{ik}(x'^l) = g^{lm}(x^l)\frac{\partial x'^i}{\partial x^l}\frac{\partial x'^k}{\partial x^m} = g^{lm}(\delta^i_l + \frac{\partial \xi^i}{\partial x^l})(\delta^l_m + \frac{\partial \xi^k}{\partial x^m}) = g^{ik}(x^l) + g^{im}\xi^k_{,m} + g^{kl}\xi^i_{,l}. \quad \text{(VII.39)}$$

On other hand, using the usual Tailor expansion we have

$$g'^{ik}(x'^l) = g^{ik}(x^l + \xi^l) = g'^{ik}(x^l) + \xi^l\frac{\partial g^{ik}}{\partial x^l} = g'^{ik}(x^l) + \xi^l g^{ik}_{,l}, \quad \text{(VII.40)}$$

hence

$$g^{ik}(x^l) + g^{im}\xi^k_{,m} + g^{kl}\xi^i_{,l} = g'^{ik}(x^l) + \xi^l g^{ik}_{,l}. \quad \text{(VII.41)}$$

We obtain that

$$g'^{ik}(x^l) = g^{ik}(x^l) - \xi^l g^{ik}_{,l} + g^{il}\xi^k_{,l} + g^{kl}\xi^i_l \ \ \textbf{or} \ \ g'^{ik} = g^{ik} + \delta g^{ik}, \quad \text{(VII.42)}$$

where

$$\delta g^{ik} = -\xi^l g^{ik}_{,l} + g^{il}\xi^k_{,l} + g^{kl}\xi^i_l. \quad \text{(VII.43)}$$

It easy to show that

$$\delta g^{ik} = g^{il}\xi^k_{;l} + g^{kl}\xi^i_{;l} \equiv \xi^{i;k} + \xi^{k;i}. \quad \text{(VII.44)}$$

Indeed,

$$\delta g^{ik} = -\xi^l(g^{ik}_{;l} - \Gamma^i_{nl}g^{nk} - \Gamma^k_{nl}g^{in}) + g^{il}(\xi^k_{;l} - \Gamma^k_{ln}\xi^n) + g^{kl}(\xi^i_{;l} - \Gamma^i_{ln}\xi^n) =$$

$$= \xi^l(\Gamma^i_{nl}g^{nk} + \Gamma^k_{nl}g^{in}) + g^{il}\xi^k_{;l} + g^{kl}\xi^i_{;l} - \xi^n(\Gamma^k_{ln}g^{il} + \Gamma^i_{ln}g^{kl}) =$$

$$= \xi^l(\Gamma^i_{nl}g^{nk} + \Gamma^k_{nl}g^{in} - \Gamma^k_{nl}g^{in} - \Gamma^i_{nl}g^{kn}) + g^{il}\xi^k_{;l} + g^{kl}\xi^i_{;l} =$$

$$= g^{il}\xi^k_{;l} + g^{kl}\xi^i_{;l} \equiv \xi^{i;k} + \xi^{k;i}. \quad \text{(VII.45)}$$

Now we know what is the variation of the contravariant metric tensor under infinitesimally small translation. If we substitute this variation into Eq.(VII.38), we obtain

$$\delta S_m = \int A_{ik}(\xi^{i;\,k} + \xi^{k;\,i})\sqrt{-g}d\Omega. \quad \text{(VII.46)}$$

From the definition of A_{ik} follows that it is a symmetric tensor. From the fact that S_m is scalar follows that the variation of S_m under translation (which is the sort of transformation of coordinates) is equal to zero, hence, we obtain

$$0 = \int A_{ik}\xi^{i;\,k}\sqrt{-g}d\Omega = \int (A^k_i\xi^i)_{;\,k}\sqrt{-g}d\Omega - \int A^k_{i;\,k}\xi^i\sqrt{-g}d\Omega. \quad \text{(VII.47)}$$

The first term in the last expression can be written as

$$(A^k_i\xi^i)_{;\,k}\sqrt{-g} = \sqrt{-g}A^k_{;k}, \ \ \text{where} \ \ A^k = A^k_i\xi^i. \quad \text{(VII.48)}$$

As follows from Eq. (VII.26)

$$\sqrt{-g}A^k_{;\,k} = (\sqrt{-g}A^k)_{,k}, \tag{VII.49}$$

and gives zero contribution to the variation. As a result we obtain that

$$\int A^k_{i;\,k}\xi^i\sqrt{-g}d\Omega = 0 \tag{VII.50}$$

nd because of arbitrariness of ξ^i we conclude that

$$A^k_{i;\,k} = 0. \tag{VII.51}$$

Taking into account that the covariant divergence of the stress-energy tensor T^i_k (see the previous lecture) is also equal to zero, one can identify A^k_i with the physical stress energy tensor within a constant factors, β and Λ:

$$A^i_k = \beta(T^i_k + \Lambda\delta^i_k). \tag{VII.52}$$

E. The final EFEs

Finally, from the principle of least action we have

$$\delta(S_g + S_m) = 0, \tag{VII.53}$$

or

$$\int \left[\alpha\left(R_{ik} - \frac{1}{2}g_{ik}R\right) + \beta(T_{(phys)ik} + \Lambda g_{ik})\right]\delta g^{ik}\sqrt{-g}d\Omega = 0. \tag{VII.54}$$

Taking into account the arbitrariness of δ and dropping label "(phys)" and putting $\Lambda = 0$ [because discussion of this famous Λ-terms is out of the scope of this course] we obtain

$$R_{ik} - \frac{1}{2}g_{ik}R = \kappa T_{ik}, \tag{VII.55}$$

where

$$\kappa = -\frac{\beta}{\alpha}. \tag{VII.56}$$

The value of κ called the Eistein constant, can be easily obtained from the weak field and slow motion limit. As we will see later

$$\kappa = \frac{8\pi G}{c^4}. \tag{VII.57}$$

This is the end of the rigorous derivation of the EFEs.
One can see that the EFEs can be rewritten in mixed components as

$$R^i_k - \frac{1}{2}\delta^i_k R = \kappa T^i_k. \tag{VII.58}$$

Contracting indices one can obtain

$$R - \frac{1}{2}4R = \kappa T, \;\; R = -\kappa T, \;\; T = T^i_k. \tag{VII.59}$$

Hence

$$R_{ik} = \kappa(T_{ik} - \frac{1}{2}g_{ik}T). \tag{VII.60}$$

In empty space-time

$$T^i_k = 0, \text{ hence } R^i_k = 0. \tag{VII.61}$$

However, it could happen that

$$R_{iklm} \neq 0. \tag{VII.62}$$

The tidal accelerations related with non zero components of the Riemann tensor in empty space are produced by gravitational waves. From

$$T^i_{;i} = 0 \tag{VII.63}$$

follows that

$$(R^i_k - \frac{1}{2}\delta^i_k R)_{;i} = R^i_{k;i} - \frac{1}{2}R_{,k} = 0. \tag{VII.64}$$

This is actually the case as it follows from the Bianchi identity. And vice versa, from pure geometrical Bianchi identity one can obtain the full description of motion of all forms of matter and fields. This means that the EFEs is complete and self-consistent description of the interaction between matter and geometry, i.e. gravitational field.

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Lecture 8. *Last updated 07.03.10*

VIII. SOLVING EFES

A. Weak field and slow motion approximation

In small velocity approximation

$$T_i^k \approx \rho c^2 u_i u^k, \tag{VIII.1}$$

where ρ is the mass density, i.e., $T_0^0 = \rho c^2$ and all other components are small, i.e., $|T_\alpha^0| \ll T_0^0$ and $|T_\alpha^\beta| \ll T_0^0$. This means that $T \equiv T_i^i \approx T_0^0$.
In weak field approximation one can neglect by the non-linear part in the Ricci tensor:

$$R_{00} = R_0^0 \approx \Gamma^\alpha_{00,\alpha} = -\frac{1}{2}\eta^{\alpha\beta} g_{00,\alpha,\beta} = \frac{1}{c^2}\phi_{,\alpha,\beta}, \tag{VIII.2}$$

where ϕ is defined by

$$g_{00} = 1 - \frac{2\phi}{c^2}. \tag{VIII.3}$$

Following usual notations

$$\eta^{\alpha\beta} g_{00,\alpha,\beta} = \triangle g_{00}, \tag{VIII.4}$$

where $\triangle$ is the Laplace operator. From EFEs we obtain

$$R_0^0 = \frac{1}{c^2}\triangle\phi = \frac{8\pi G}{c^4}(T_0^0 - \frac{1}{2}T) \approx \frac{8\pi G}{c^4}(T_0^0 - \frac{1}{2}T_0^0) = \frac{4\pi G}{c^4}T_0^0. \tag{VIII.5}$$

Hence,

$$\triangle\phi = 4\pi G\rho. \tag{VIII.6}$$

This is the Poisson equation, hence, as one can see, in this approximation EFEs give the Newtonian gravity and ϕ is the Newtonian gravitational potential.

B. The Schwarzschild metric as an exact solution of EFEs

Let r, θ, ϕ are spherical space coordinates. The most general spherically symmetric gravitational field can be described by the interval in the following form

$$ds^2 = h(r,t)dr^2 + k(r,t)(sin^2\theta d\phi^2 + d\theta^2) + l(r,t)dt^2 + a(r,t)drdt. \qquad \text{(VIII.7)}$$

By transformations of coordinates

$$r = f_1(r', t'), t = f_2(r', t'), \qquad \text{(VIII.8)}$$

we always can make

$$a(r,t) = 0 \ \text{ and } \ k(r,t) = -r^2. \qquad \text{(VIII.9)}$$

Thus

$$ds^2 = e^\nu c^2 dt^2 - r^2(sin^2\theta d\phi^2 + d\theta^2) - e^\lambda dr^2. \qquad \text{(VIII.10)}$$

Taking into account that

$$g_{00} > 0 \ \text{ and } \ g_{11} < 0, \qquad \text{(VIII.11)}$$

we can see that

$$g_{00} = e^\nu, \ g_{11} = -e^\lambda, \ g_{22} = -r^2, \ \text{ and } \ g_{33} = -r^2 \sin^2\theta \qquad \text{(VIII.12)}$$

$$g^{00} = e^{-\nu}, \ g^{11} = -e^{-\lambda}, \ g^{22} = -r^{-2} \ \text{ and } \ g^{33} = -r^{-2}\sin^{-2}\theta. \qquad \text{(VIII.13)}$$

Now we can calculate the Christoffell symbols:

$$\Gamma^1_{11} = \frac{\lambda'}{2}, \ \Gamma^0_{10} = \frac{\nu'}{2}, \ \Gamma^2_{33} = -\sin\theta\cos\theta, \ \Gamma^0_{11} = \frac{\lambda}{2}e^{\lambda-\nu}, \qquad \text{(VIII.14)}$$

$$\Gamma^1_{22} = -re^{-\lambda}, \ \Gamma^1_{00} = \frac{\nu}{2}e^{\nu-\lambda}, \ \Gamma^2_{12} = \Gamma^3_{13} = \frac{1}{r}, \ \Gamma^3_{23} = \cot\theta, \qquad \text{(VIII.15)}$$

$$\Gamma^0_{00} = \frac{\dot\nu}{2}, \ \Gamma^1_{10} = \frac{\dot\lambda'}{2}, \ \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \qquad \text{(VIII.16)}$$

where $'$ means partial derivative with respect to r. Then after straightforward calculations of the components of the Ricci tensor we obtain the Einstein's equations:

$$\frac{8\pi G}{c^4}T^1_1 = -e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) + \frac{1}{r^2}, \qquad \text{(VIII.17)}$$

$$\frac{8\pi G}{c^4}T^2_2 = \frac{8\pi G}{c^4}T^3_3 =$$

$$= -\frac{1}{2}e^{-\lambda}\left(\nu'' + \frac{\nu'^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) + \frac{1}{2}e^{-\nu}\left(\ddot\lambda + \frac{\dot\lambda^2}{2} - \frac{\dot\lambda\dot\nu}{2}\right), \qquad \text{(VIII.18)}$$

$$\frac{8\pi G}{c^4}T^0_0 = -e^{-\lambda}\left(\frac{1}{r^2} - \frac{\lambda'}{r}\right) + \frac{1}{r^2}, \qquad \text{(VIII.19)}$$

$$\frac{8\pi G}{c^4}T^1_0 = -e^{-\lambda}\frac{\dot\lambda}{r}. \qquad \text{(VIII.20)}$$

In vacuum, where all $T^i_k = 0$, we have

$$- e^{-\lambda} \left(\frac{\nu'}{r} + \frac{1}{r^2} \right) + \frac{1}{r^2} = 0, \tag{VIII.21}$$

$$e^{-\lambda} \left(\frac{\lambda'}{r} - \frac{1}{r^2} \right) + \frac{1}{r^2} = 0, \tag{VIII.22}$$

$$\dot{\lambda} = 0, \tag{VIII.23}$$

The most unpleasant equation fortunately is not independent and follows from other three equations. One can proove this by straightforward calculations or by using the Bianchi identity. From equation (VIII.23) follows that $\lambda = \lambda(r)$, i.e. does not depend on t. From equations (VIII.21) and (VIII.22) follows that

$$\lambda' + \nu' = 0, \text{ hence } \lambda + \nu = f(t). \tag{VIII.24}$$

Now we can use our last freedom in coordinate transformation, namely we can transform the time coordinate, $t = f(t')$ to make $f(t) = 0$. As a result we obtain

$$e^{-\lambda} = e^{\nu}. \tag{VIII.25}$$

Thus we actually proved a very important theorem: If a gravitational field is spherical symmetric then this field is static! Now the system has been reduced to the single equation (VIII.22), which after multiplying by r^2 can be written as

$$e^{-\lambda} (r\lambda' - 1) + 1 = 0 \text{ or } - \left(e^{-\lambda} r\right)' + 1 = 0. \tag{VIII.26}$$

Finally

$$e^{-\lambda} = e^{\nu} = 1 + \frac{A}{r}, \tag{VIII.27}$$

where A is a constant of integration. One can see that if $r \to \infty$, then

$$e^{-\lambda} = e^{\nu} \to 1, \tag{VIII.28}$$

which corresponds to the Minkowskian space-time.
In order to determine the constant A let consider a test particle far from the centre of gravitating object. It's radial acceleration is given by the geodesic equation:

$$\frac{d^2 r}{ds^2} + \Gamma^1_{ik} u^i u^k = 0. \tag{VIII.29}$$

If we assume that the particle moves slowly, i.e. four-velocity $u^i \approx \delta^i_0$ and $ds \approx cdt$ we obtain

$$\frac{d^2 r}{dt^2} \approx -c^2 \Gamma^1_{ik} \delta^i_0 \delta^k_0 = -c^2 \Gamma^1_{00} =$$

$$= -\frac{c^2}{2} g^{1n} (g_{0n,0} + g_{n0,0} - g_{00,n}) = -\frac{c^2}{2} g^{11} (g_{01,0} + g_{10,0} - g_{00,1}) \approx -\frac{c^2}{2} \frac{dg_{00}}{dr} =$$

$$= -\frac{c^2}{2} \frac{de^{-\lambda}}{dr} = -\frac{c^2}{2} \frac{d}{dr} (1 + \frac{A}{r}) = \frac{Ac^2}{2r^2}. \tag{VIII.30}$$

On other hand we know from Newtonian theory that

$$\frac{d^2r}{dt^2} = -\frac{GM}{r^2}, \tag{VIII.31}$$

hence the constant of integration

$$A = -\frac{2Gm}{c^2} = -r_g \text{ and } g_{00} = 1 - \frac{r_g}{r}, \tag{VIII.32}$$

where r_g is the so called gravitational radius

$$r_g = \frac{2Gm}{c^2}. \tag{VIII.33}$$

Finally we derived the famous solution of the EFEs obtained by K. Schwarzschild in 1916, the same year when Einstein published his equations. This solution is called the Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - r^2 (sin^2\theta d\phi^2 + d\theta^2) - \frac{dr^2}{1 - \frac{r_g}{r}}. \tag{VIII.34}$$

One can see that this metric describes a curved space-time. To prove, for example, that even the space itself is curved, let us compare the physical radial distance, l, with the corresponding circumference, C. In the flat Euclidian space

$$l = \frac{C}{2\pi}, \tag{VIII.35}$$

while in the case of the Schwarzschild metric

$$dl^2 = \frac{dr^2}{1 - \frac{r_g}{r}} + r^2 (\sin^2\theta d\phi^2 + d\theta^2), \tag{VIII.36}$$

hence

$$l = \int_{r_1}^{r_2} \frac{dr}{\sqrt{1 - \frac{r_g}{r}}} > r_2 - r_1 = \frac{l_{circl_2} - l_{circl_1}}{2\pi}. \tag{VIII.37}$$

One can see also that time runs at a different rate at different radii, indeed

$$d\tau = \sqrt{g_{00}} dt = \sqrt{1 - \frac{r_g}{r}} dt. \tag{VIII.38}$$

C. Physical singularity versus coordinate singularity in the Schwarzschild metric

We can prove that there is no physical singularity at $r = r_g$. For that we produce the following transformation of coordinates

$$c\tau = \pm ct \pm \int \frac{f(r)dr}{1 - \frac{r_g}{r}}, \tag{VIII.39}$$

$$R = ct + \int \frac{dr}{\left(1 - \frac{r_g}{r}\right) f(r)}, \tag{VIII.40}$$

where $f(r)$ is an arbitrary function. Now the interval can be written in the following form:

$$ds^2 = \frac{1 - \frac{r_g}{r}}{1 - f^2} (c^2 d\tau^2 - f^2 dR) - r^2 (d\theta^2 + sin^2\theta d\phi^2). \tag{VIII.41}$$

To eliminate "singularity" at $r = r_g$, we can choose $f(r)$ in such a way that $f(r_g) = 1$. For example,

$$f(r) = \sqrt{\frac{r_g}{r}}. \tag{VIII.42}$$

In this case

$$R - c\tau = \int \frac{(1 - f^2)dr}{\left(1 - \frac{r_g}{r}\right) f} = \int \sqrt{\frac{r}{r_g}} dr = \frac{2}{3} \frac{r^{3/2}}{r_g^{1/2}} \tag{VIII.43}$$

and

$$r = \frac{3}{2}(R - c\tau)^{2/3} r_g^{1/3}, \tag{VIII.44}$$

$$ds^2 = c^2 d\tau^2 - \frac{dR^2}{\left[\frac{3}{2r_g}(R - c\tau)\right]^{2/3}} - \left[\frac{3}{2}(R - c\tau)\right]^{4/3} r_g^{2/3}(d\theta^2 + sin^2\theta d\varphi^2). \tag{VIII.45}$$

We can see that there is now singularity at $r = r_g$, indeed if $r = r_g$

$$\frac{3}{2}(R - c\tau) = r_g. \tag{VIII.46}$$

In other words, the formal "singularity" ar $r = r_g$ can be removed by the transformation of coordinates. The real physical singularity does take place at $r = 0$ when, say, the scalar curvature is infinite (one can easily verify this by straightforward calculations) and this fact can not be removed by any transformation of coordinates.

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Lecture 9. *Last updated 07.03.10*

IX. BLACK HOLES

A. Limit of stationarity

Let us consider ds for the test particle in rest, i.e. put $dr = d\theta = d\phi = 0$, in this case

$$ds^2 = g_{00} dx^{0^2}, \tag{IX.1}$$

If $g_{00} = 0$ then $ds^2 = 0$, which means that the world line of the particle at rest is the world line of light, hence at the surface $g_{00} = 0$ no particle with finite rest mass can be at rest. Thus the surface $g_{00} = 0$ is called the limit of stationarity.

B. Event horizon

Let us consider a surface

$$F(r) = const \tag{IX.2}$$

and let

$$n_i = F_{,i} \tag{IX.3}$$

is its normal. If $g^{11} = 0$ then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = 0, \tag{IX.4}$$

which means that n_i is the null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

C. Schwarzschild black holes

Schwarzschild Black holes are described by the following metric

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2 \theta d\phi^2 + d\theta^2\right), \tag{IX.5}$$

obtained in the previous lecture.One can see that both the limit of stationarity and the event horizon are located at $r = r_g$.
Let us consider the structure of light cone in the Schwarzschild metric using the new coordinates $c\tau$ and R introduced in Lecture 8. Putting $ds = 0$, we have

$$c\frac{d\tau}{dR} = \pm \frac{1}{\left(\frac{3}{2r_g}(R - c\tau)\right)^{1/3}} = \pm\sqrt{\frac{r_g}{r}}. \tag{IX.6}$$

Thus we can see that if $r > r_g$

$$|c\frac{d\tau}{dR}| < 1 \tag{IX.7}$$

and the surface $r = const$ is inside the light cone, while for $r < r_g$

$$|c\frac{d\tau}{dR}| > 1 \tag{IX.8}$$

and the surface $r = const$ is outside the light cone, which means that all particles and even photons should propagate inward. In order words we can see that the surface $r = r_g$ is the event horizon.

D. Kerr Black Holes

The Kerr metric describing the gravitational field of rotating black holes has the following form

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2 +$$

$$+ \frac{2r_g r a}{\rho^2}\sin^2\theta c d\phi dt, \tag{IX.9}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta, \;\; \Delta = r^2 - r_g r + a^2 \;\text{ and }\; a = \frac{J}{mc} \tag{IX.10}$$

and J is the specific angular momentum of the black hole.

1. *Limit of stationarity*

The location of the limit of stationarity, r_{st}, corresponding to $g_{00} = 0$, in the Kerr metric is determined from the equation

$$1 - \frac{r_g r}{\rho^2} = 0, \;\text{ thus }\; r^2 - r_g r + a^2\cos^2\theta = 0. \tag{IX.11}$$

Solving this equation we obtain that

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}. \tag{IX.12}$$

2. *Event horizon*

The location of the event horizon, r_{hor} is determined by $g^{11} = 0$. In the Kerr metric this corresponds to $g_{11} = \infty$, i.e. corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0, \tag{IX.13}$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2 \cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2 \cos^2\theta}, \tag{IX.14}$$

hence

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}. \tag{IX.15}$$

E. "Ergosphere" and Penrose process

1. *Ergosphere*

The region between the limit of stationarity and the event horizon is called the "ergosphere". By the Penrose process it is possible to extract the rotational energy of the Kerr black hole.

2. *Penrose process*

The Penrose process is a process wherein energy can be extracted from a rotating black hole. That extraction is made possible because the rotational energy of the black hole is located not inside the event horizon, but outside in a curl gravitational field. Such field is also called gravi-magnetic field. All objects in the ergosphere are unavoidably dragged by the rotating spacetime. Imagine that some body enters into the black hole and then it is split there into two pieces. The momentum of the two pieces of matter can be arranged so that one piece escapes to infinity, whilst the other falls past the outer event horizon into the black hole. The escaping piece of matter can have a greater mass-energy than the original infalling piece of matter. In other words, the captured piece has negative mass-energy. The Penrose process results in a decrease in the angular momentum of the black hole, and that reduction corresponds to a transference of energy whereby the momentum lost is converted to energy extracted. As a result of the Penrose process a rotating black hole can eventually lose all of its angular momentum, becoming a non-rotating (i.e. the Schwarzschild) black hole.

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Lecture 10. *Last updated 07.03.10*

X. IN VICINITY OF THE SCHWARZSCHILD BLACK HOLE

A. Test particles in the Schwarzschild Metric

B. Stable and Unstable Circular Orbits

C. Propagation of light in the Schwarzschild metric

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Lecture 11. *Last updated 07.03.10*

XI. EXPERIMENTAL CONFIRMATION OF GR AND GRAVITATIONAL WAVES (GWS)

A. Relativistic experiments in the Solar system and Binary pulsar

B. Propagation of GWs

C. Detection of GWs

D. Generation of GWs

E. Examples, problems and summary

XII. SUMMARY OF THE COURSE

i)
ii)
iii)
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