DIFFERENTIAL EQUATIONS

Definition: Given some function $y(x)$ of x, a differential equation (DE) is any equation involving derivatives of y w.r.t. x.

The order of the DE is the ordinality of the highest derivative:

Examples:

$$\frac{dy}{dx} + \cos x \, y(x) = 0 \qquad 1^{st} \text{ order}$$

$$\frac{d^2y}{dx^2} + e^x = 0 \qquad 2^{nd} \text{ order}$$

$$\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = 0 \qquad 2^{nd} \text{ order}$$

1^{st} order DE's, Direct Integration:

Example: $\frac{dy}{dx} = 1$

$$\int dy = \int dx$$

$$y + c_y = x + c_x$$

constants of integration

So $\underline{y = x + c}$

1^{st} order DE's have one freely-chosen constant, called constant of integration, to be determined by a boundary condition.

E.g. $y(x=0) = 1$, then $y = c + 1$

Example: $\frac{dy}{dx} = 2y \quad , \quad y(0) = 1$

$$\frac{1}{2} \frac{dy}{y} = dx$$

$$\frac{1}{2} \int \frac{dy}{y} = \int dx$$

$$\therefore \frac{1}{2} \ln y = x + c_1$$

$$\therefore \ln y = 2x + c_2$$

$$\therefore e^{\ln y} = e^{2x + c_2}$$

$$\therefore y = e^{2c_2} e^{2x} = c_3 e^{2x}$$

Boundary condition : $x = 0 \Rightarrow y = 1$

So $1 = c_3$

and $\underline{\underline{y = e^{2x}}}$

1^{st} order by separation of variables:

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}$$

Then $\underline{\int g(y)\,dy = \int f(x)\,dx}$

Example: $\frac{dy}{dx} = (1+x)(1+y)$

i.e. $\int \frac{dy}{1+y} = \int (1+x)\,dx = x^2 + \frac{x^2}{2} + C$

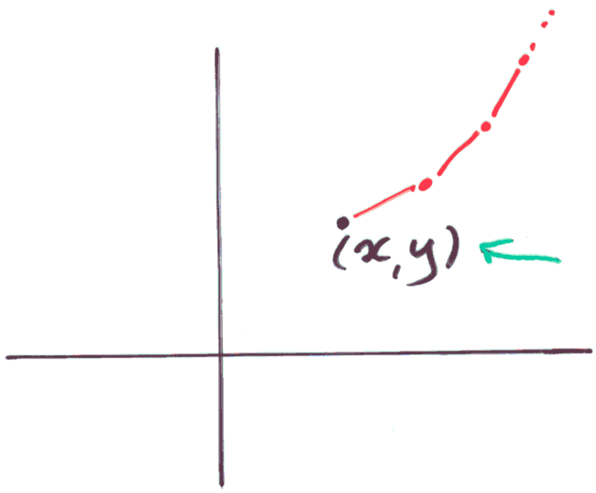
so $\ln(1+y) = x^2 + \frac{1}{2}x^2 + C$

$\therefore 1+y = e^{x^2 + \frac{1}{2}x^2 + C}$

$$\underline{y = C e^{x^2 + \frac{1}{2}x^2} - 1}$$

1^{st} order General Case:

$$\frac{dy}{dx} = f(x,y)$$



• Knowing this point is boundary condition (b.c.)

Iterate:

$x \to x + dx$

Calculate $dy = f(x,y)\,dx$

$y \to y + dy$

Loop

Linear 1st order Analytic solutions:

$$p(x)\frac{dy}{dx} + q(x)y(x) = f(x)$$

(a) Homogeneous case, $f(x) = 0$

then $\frac{1}{y}\frac{dy}{dx} = -\frac{q}{p}$

so $\int \frac{dy}{y} = -\int \frac{q}{p}\,dx$

$\therefore \ln y = -\int \frac{q}{p}\,dx + C$

or $\underline{y = Ce^{-\int q/p\,dx}}$ is the solution

Example: $\frac{dy}{dx} + xy = 0$

$$y = Ce^{-\int x\,dx} = \underline{\underline{Ce^{-\frac{1}{2}x^2}}}$$

Example: $x\frac{dy}{dx} + y = 0$

$$y = Ce^{-\int \frac{dx}{x}} = Ce^{-\ln x} = \underline{\underline{\frac{C}{x}}}$$

(b) Inhomogeneous case, $f(x) \neq 0$

We solve this by

(i) Find a "particular solution" $y_p(x)$, i.e. a solution without a constant of integration

(ii) Find the "complementary function", $y_c(x)$, i.e. a solution of the homogeneous equation.

(iii) Then the general solution is $y(x) = y_p(x) + y_c(x)$

OK – but how to find $y_p(x)$?

(i.b): Variation of the constant:

$y_c = Ce^{-\int q/p \, dx}$. Try putting $u(x)$ for C.

Example: $\frac{dy}{dx} - 2xy = x - x^3$

First solve $\frac{dy}{dx} - 2xy = 0$ for y_c

(see above) $\underline{y_c(x) = Ce^{x^2}}$

For $y_p(x)$ try $u(x)e^{x^2}$

Then $\frac{d}{dx}\left(u(x)e^{x^2}\right) - 2x\,u(x)e^{+x^2} = x - x^3$

i.e. $\frac{du}{dx}e^{x^2} + u.2xe^{x^2} - 2xue^{x^2} = x - x^3$

so $\frac{du}{dx} = (x - x^3)e^{-x^2}$

$\Rightarrow u(x) = \frac{1}{2}x^2e^{-x^2}$

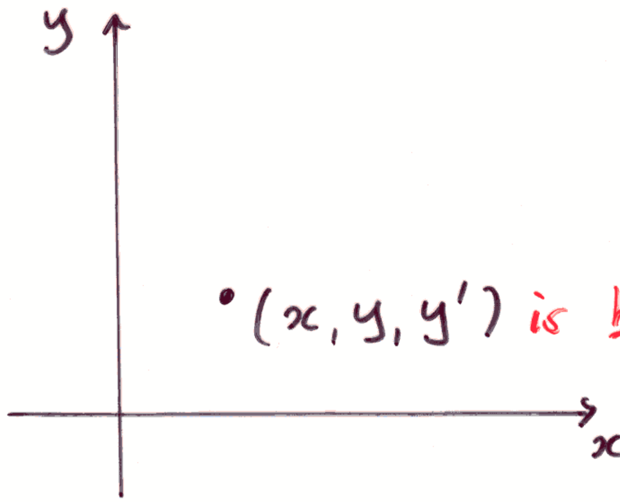
So $y_p(x) = u(x)e^{x^2} = \frac{1}{2}x^2$

And $y = y_p + y_c = \underline{\underline{Ce^{x^2} + \frac{1}{2}x^2}}$

is the general solution.

2^{nd} order General Case

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$



(x, y, y') is boundary condition

Iterate:

Knowing x, y, y' calculate $\frac{d^2y}{dx^2}$

Move dx

Calculate new $y \rightarrow y + y'dx$

Calculate new $y' \rightarrow y' + y''dx$

Loop

NOTE 2 b.c.'s

$\Rightarrow$ 2 constants of integration

Example $\dfrac{d^2y}{dx^2} = 0$

General solution is $y(x) = ax + b$

constants of integration

Given boundary conditions,

e.g. $\left.\dfrac{dy}{dx}\right|_{x=1} = 6$

$\left.y\right|_{x=0} = 1$

We obtain $\underline{a = 6\ ,\ b = 1}$

Or, e.g. $\left.y\right|_{x=0} = 1$

$\left.y\right|_{x=10} = 61$

Again, $\underline{a = 6,\ b = 1}$

Linear 2nd-order D.E.'s

$$p(x)y'' + q(x)y' + r(x)y = f(x)$$

(a) Homogenous case, $f(x) = 0$

Example: $p(x) = a$
$q(x) = b$
$r(x) = c$

Try $y = e^{\lambda x}$

Then $a\lambda^2 e^{\lambda x} + b\lambda e^{\lambda x} + c e^{\lambda x} = 0$

so $a\lambda^2 + b\lambda + c = 0$

i.e. $\lambda_{(\pm)} = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

and so the general solution is

$$\underline{y(x) = c_1 e^{\lambda_{(+)} x} + c_2 e^{\lambda_{(-)} x}}$$

c_1 and c_2 are constants of integration, to be determined from boundary conditions.

Example Oscillatory Motion

Hooke's Law: $F = -kx = m\frac{d^2x}{dt}$
+ $F = ma$

i.e. $\frac{d^2x}{dt^2} + \frac{k}{m}x = 0$

so that
$$p(x) = a = 1$$
$$q(x) = b = 0$$
$$r(x) = c = k/m$$
$$f(x) = 0$$

$$\lambda_{(\pm)} = \frac{\pm\sqrt{-4k/m}}{2} = \pm\sqrt{-\frac{k}{m}}$$
$$= \pm i\sqrt{k/m} = \pm i\omega$$

General solution is
$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}$$
OR
$$= A\cos\omega t + B\sin\omega t = C\cos(\omega t + \varphi)$$
OR
$$= A_{real}\, e^{i(\omega t + \varphi)}$$
OR
$$= A_{complex}\, e^{i\omega t}$$
ETC

Example: Damped Oscillatory Motion

Add in a velocity term, in $\dot{x} = \frac{dx}{dt}$

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0^2 x(t) = 0$$

$x = e^{\lambda t}$ gives $\lambda^2 + 2\beta\lambda + \omega_0^2 = 0$

$$\lambda_{(\pm)} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

and general solution is

$$x(t) = c_1 e^{\lambda_{(+)} t} + c_2 e^{\lambda_{(-)} t}$$

Light Damping $\beta^2 < \omega_0^2$

$$\Rightarrow x(t) = A e^{-\beta t} \cos(\omega_0 t + \varphi)$$

= simple damped harmonic motion

Heavy Damping $\beta^2 > \omega_0^2$ $\quad \Lambda^2 = \beta^2 - \omega_0^2$

$$\Rightarrow x(t) = c_1 e^{-(\beta + \Lambda)t} + c_2 e^{-(\beta - \lambda)t}$$

= exponentially decreasing

Critical Damping $\beta^2 = \omega_0^2$ $\quad \Lambda = 0$

$$\Rightarrow x(t) = (c_1 + c_2 t) e^{-\beta t}$$

In[43]:= **soln = DSolve[x''[t] + 2 x'[t] + 401 x[t] == 0, x[t], t]**

Out[43]= $\{\{x[t] \to E^{-t}\, C[2]\, Cos[20\,t] - E^{-t}\, C[1]\, Sin[20\,t]\}\}$

In[44]:= **light = x[t] /. soln[[1]] /. C[1] -> 5 /. C[2] -> 0**

Out[44]= $-5\, E^{-t}\, Sin[20\,t]$

In[48]:= **soln = DSolve[x''[t] + 40 x'[t] + 400 x[t] == 0, x[t], t]**

Out[48]= $\{\{x[t] \to E^{-20\,t}\, C[1] + E^{-20\,t}\, t\, C[2]\}\}$

In[49]:= **crit = x[t] /. soln[[1]] /. C[1] -> 5 /. C[2] -> 0**

Out[49]= $5\, E^{-20\,t}$

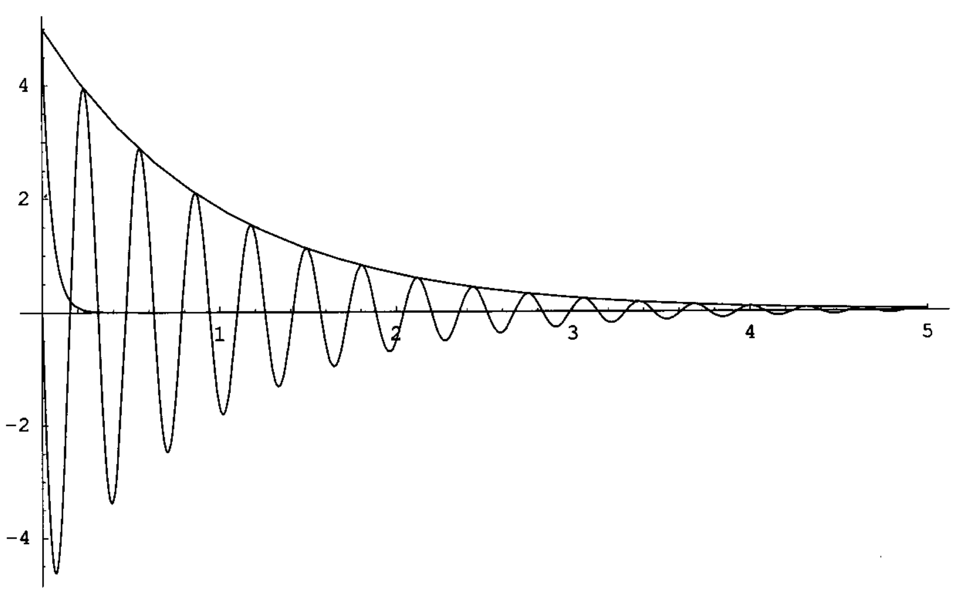
In[61]:= **soln = DSolve[x''[t] + 400 x'[t] + 400 x[t] == 0, x[t], t]**

Out[61]= $\left\{\left\{x[t] \to E^{20\,\left(-10-3\sqrt{11}\right)\,t}\, C[1] + E^{-\frac{20\,t}{10+3\sqrt{11}}}\, C[2]\right\}\right\}$

In[62]:= **heavy = x[t] /. soln[[1]] /. C[1] -> 0 /. C[2] -> 5**

Out[62]= $5\, E^{-\frac{20\,t}{10+3\sqrt{11}}}$

In[63]:= **Plot[{light, crit, heavy},**
{t, 0, 5}, PlotRange -> All]



Linear Second-Order D.E.'s

$$p(t)\ddot{x} + q(t)\dot{x} + r(t)x = f(t)$$

(b) Inhomogeneous Case $f(t) \neq 0$

Again, we look for

(i) a particular solution $x_p(t)$

(ii) a complementy function $x_c(t)$

Example Forced Oscillatory Motion

Consider $\ddot{x} + \omega_0^2 x = \cos\omega t$

Particular solution: Try $x = A\cos\omega t$

Then $-\omega^2 A\cos\omega t + \omega_0^2 A\cos\omega t = \cos\omega t$

So $A = \dfrac{1}{\omega_0^2 - \omega^2}$

Note: Amplitude $A \to \infty$ at $\omega = \omega_0$

i.e. Resonance

Phase reverses at $\omega = \omega_0$

General Solution: Add c.f.

i.e. harmonic oscillation at ω_0