Week 7 : Lecture 2,3 : Green's Functions for 1D problem

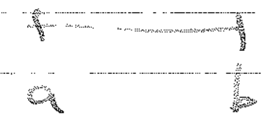
→ Consider:

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = F(x)$$

Boundary Conditions:

$y(a) = 0$

$y(b) = 0$



Suppose we have y_1 & y_2 as solutions of the homogeneous equation:
Choose them such that

$y_1(a) = 0$

$y_2(b) = 0$.

Previously, we explained that $G(x, x')$ obeys

① $L_x G(x, x') = \delta(x - x')$

Now we will specify bdy conditions

② $G(a, x') = 0$

& $G(b, x') = 0$

③ ~~$G(x, x')$~~

& Continuity of $G(x, x')$:

③ $G(x, x')$ is continuous at $x = x'$

& discontinuity of $\frac{dG}{dx}$

④ $\frac{dG(x, x')}{dx}$ has a discontinuity at $x = x'$

For $x < x'$:

$$\frac{d^2G}{dx^2} + P\frac{dG}{dx} + QG = 0.$$

$\Rightarrow G(x, x') = A(x')\, y_1(x)$

to ensure $G(a, x') = 0$

For $x > x'$:

~~you have~~ homogeneous eq.

$$G(x, x') = B(x')\, y_2(x)$$

to ensure $G(b, x') = 0$

Continuity:

$$A(x')\, y_1(x') = B(x')\, y_2(x')$$

$$\frac{A(x')}{B(x')} = \frac{y_2(x')}{y_1(x')}.$$

Discontinuity

$$\int_{x=x'-\epsilon}^{x'+\epsilon} \frac{d}{dx}\left(\frac{dG}{dx}\right) + \int_{x'-\epsilon}^{x'+\epsilon} P(x)\frac{dG}{dx}$$

$$+ \int_{x'-\epsilon}^{x'+\epsilon} Q(x)\, G\, dx$$

$$= \int \delta(x-x')\, dx$$

$$= 1$$

By continuity:

$$\int QG = 0 \quad \text{as } \epsilon \to 0.$$

$$\int P(x)\frac{dG}{dx}\,dx = \int \frac{d}{dx}(PG) - \int G\frac{dP}{dx}\,dx$$

$$= \left[PG\right]_{x'-\epsilon}^{x'+\epsilon} - \int_{x'-\epsilon}^{x'+\epsilon} G\frac{dP}{dx}\,dx$$

$\to 0$ by continuity of PG $\qquad \to$ zero.

$$\therefore \quad \left(\frac{dG}{dx}\right)_{x'+\epsilon} - \left(\frac{dG}{dx}\right)_{x'-\epsilon} = 1$$

$$\left.\frac{dG}{dx}\right|_{-} = A(x')\,y_1'(x')$$

$$\left.\frac{dG}{dx}\right|_{+} = B(x')\,y_2'(x')$$

$$\Rightarrow \quad B(x')\,y_2'(x') - A(x')\,y_1'(x') = 1$$

$$\Rightarrow \quad B y_2' - A y_1' = 1$$

$$A y_1 = B y_2$$

$$\Rightarrow \quad \left(\frac{A y_1}{y_2}\right) y_2' - A y_1' = 1$$

$$\Rightarrow \quad A\,(y_1 y_2' - y_2 y_1') = y_2$$

$$\Rightarrow \quad A = \frac{y_2}{(y_1 y_2' - y_2 y_1')} = \frac{y_2}{W(y_1, y_2)}$$

$$B = \frac{y_1}{W(y_1, y_2)}$$

$$A(x') = \frac{y_2(x')}{W(x')}$$

$$B(x') = \frac{y_1(x')}{W(x')}$$

$$W(x') = y_1(x')\, y_2'(x') - y_1'(x')\, y_2(x')$$

$$\therefore\ G(x,x') = \begin{cases} \dfrac{y_2(x')\, y_1(x)}{W(y_1(x'), y_2(x'))} & x < x' \\[2ex] \dfrac{y_1(x')\ y_2(x)}{W(y_1(x'), y_2(x'))} & x > x' \end{cases}$$

$$\Rightarrow y(x) = \int_a^b G(x,x')\, F(x')\, dx'$$

$$= y_1(x) \int_x^b \frac{y_2(x')F(x')}{W(x')}\, dx'$$

$$+ y_2(x) \int_a^x \frac{y_1(x')F(x')}{W(x')}\, dx'$$

Example:

$$\frac{d^2y}{dx^2} + y = \operatorname{Cosec} x.$$

in the interval: $\left[0, \frac{\pi}{2}\right]$

Bdy Conditions: $y(0) = 0$

$y\left(\frac{\pi}{2}\right) = 0$

Homogeneous of:

$$\frac{d^2y}{dx^2} = -y$$

$\Rightarrow$ $\begin{cases} \operatorname{Cos} x \\ \operatorname{Sin} x \end{cases}$

$y_1(0) = 0$

$\Rightarrow y_1(x) = \operatorname{Sin} x$

$y_2\left(\frac{\pi}{2}\right) = 0$

$\Rightarrow y_2(x) = \operatorname{Cos} x.$

$y_c(x) =$

Solution :-

$$y(x) = y_1(x) \int_x^{\pi/2} \frac{y_2(x') F(x')}{W(x')} dx' + y_2(x) \int_0^x \frac{y_1(x') F(x')}{W(x')} dx'$$

$$W(x') = y_1 y_2' - y_1' y_2$$

$$= \sin x' . (-\sin x') - \cos x' \cos x'$$

$$= -1$$

$$y(x) = \sin x . \int_x^{\pi/2} \frac{\cos(x')}{-1 . \sin(x')} dx' + \cos x \int_0^x \frac{(\sin x')}{-1 . \sin x'} dx'$$

$$= -\sin x \int_x^{\pi/2} \cot (x') dx'$$

$$- \cos x \int_0^x dx'$$

$\int \cot x \, dx$

$= \int \frac{\cos x}{\sin x} \, dx$

$= \int \frac{+\frac{d}{dx}(\sin x)}{\sin x} \, dx$

$= \int \frac{d}{dx} \ln(\sin x) \, dx$

$= \ln(\sin x).$

$\therefore y(x) = -\sin x \left[\ln(\sin x) \right]_{x}^{\pi/2}$

$-x \ldots$

$= -\sin x \times 0 + \sin x \cdot \ln(\sin x)$

$- x \cos x$

$= \underline{(\sin x) \ln(\sin x)} - x \cos x.$

String vibrating between 2 fixed points.

$$-y'' + \ddot{y} = F(x)\, e^{i\omega t}$$

Try $y(x,t) = u(x)\, e^{i\omega t}$

$$y'' = u''\, e^{i\omega t}$$

$$\ddot{y} = -\omega^2 u\, e^{i\omega t}$$

$$\therefore\ -u'' e^{i\omega t} - \omega^2 u\, e^{i\omega t} = F(x)\, e^{i\omega t}$$

$$\Rightarrow \boxed{u'' + \omega^2 u = F(x)}$$

That is the equation we have solved

→ A Green's function problem
that cropped up in
electrodynamics.

$$-k^2 G(t,t') - \frac{\partial^2 G(t,t')}{\partial t^2} = \delta(t-t')$$

$$G(t,t') = \int_{-\infty}^{\infty} \tilde{G}(\omega,t')\, e^{i\omega t}\, d\omega$$

$$\tilde{G}(\omega,t') = \int_{-\infty}^{\infty} G(t,t')\, e^{-i\omega t}\, \frac{dt}{2\pi}$$

$$\int_{-\infty}^{\infty} d\omega\; e^{i\omega t}\left(-k^2 \tilde{G}(\omega,t') + \omega^2 \tilde{G}(\omega,t')\right)$$

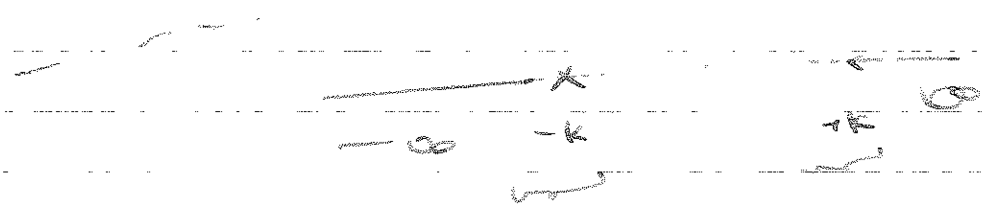
$$= \int e^{i\omega(t-t')}\, \frac{d\omega}{2\pi}$$

$$\therefore\; \tilde{G}(\omega,t') = \frac{e^{i\omega t'}}{(\omega^2-k^2)}$$

→ $G(t, t')$

$$= \int_{-\infty}^{\infty} \frac{e^{-i\omega t'}}{(\omega^2 - k^2)} \cdot e^{i\omega t} \, d\omega$$

→ integral over



→ contour deformation

Diff. bdy conditions :

effect at a time t should come after cause rather than before,