Partial Differential Equations (PDE).

Previously we studied methods of solving (in the main) linear first and second order ordinary differential equations (ODE) - that is equations with $y(x)$ only dependent on a single variable x.

Partial differential equations (PDE's) are equations involving functions of several variable and their partial derivatives. As such, PDE's play a central role in mathematical Physics.

Examples:

$$\frac{\partial \psi(x,t)}{\partial x} = \frac{1}{c}\frac{\partial \psi(x,t)}{\partial t}$$

linear PDE
1^{st} order

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

Laplace Equation in 3-d
2^{nd} order PDE

$$\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

wave equation in 1-d.
(2^{nd} order, linear PDE)

There are several techniques one can use in an attempt to solve PDE's. We shall consider only 1st and second order PDE's in this course.

1st order PDE.

Lets look at a simple example:

$$\frac{\partial \psi(x,t)}{\partial x} - \frac{1}{a}\frac{\partial \psi(x,t)}{\partial t} = 0 .$$

$\psi = \psi(x,t)$, depends on 2 independent variables x, t and a is a constant.

Consider change of variables:

$$x' = x - at \quad ; \quad t' = x + at .$$

Then $\dfrac{\partial}{\partial x'} = \dfrac{\partial x}{\partial x'}\dfrac{\partial}{\partial x} + \dfrac{\partial t}{\partial x'}\dfrac{\partial}{\partial t}$

$\dfrac{\partial x'}{\partial x} = 1, \quad \dfrac{\partial x'}{\partial t} = -a \qquad \therefore \quad \dfrac{\partial}{\partial x'} = \dfrac{\partial}{\partial x} - \dfrac{1}{a}\dfrac{\partial}{\partial t}$

Our Equation can therefore be written as:-

$$\boxed{\frac{\partial}{\partial x'}\psi(x',t') = 0}$$

Now an equation like $\frac{df(x)}{dx} = 0$ has solution

$f(x) = \text{constant}$. But when $\psi(x', t')$ depends on 2

variables, $\frac{\partial}{\partial x'} \psi(x', t') = 0 \Rightarrow \psi(x', t') = f(t')$.

(Since $\frac{\partial}{\partial x'} f(t') = 0$ as t', x' are independent).

But $t' = x + at$.

Hence general solution of $\frac{\partial \psi}{\partial x} - \frac{1}{a}\frac{\partial \psi}{\partial t} = 0$

is $\psi = \psi(x+at)$ – i.e. only depends on

x and t through the combination $\underline{x+at}$.

General 1st order PDE's.

$$\frac{\partial \psi}{\partial x}(x,y) + P(x,y)\frac{\partial \psi}{\partial y}(x,y) = \varphi(x,y)$$

this is linear since ψ appears at most as linear.

Look at some simple cases:–

$\varphi = 0$; $P(x,y) = f(x)\, g(y)$. (factorized).

look for factorized solution with seperation of variables:

$$\psi(x,y) = u(x)\, v(y).$$

$$\frac{\partial \psi}{\partial x} = \left(\frac{\partial u}{\partial x}\right) v(y) + u(x) \cancelto{0}{\frac{\partial v(y)}{\partial x}}$$

$$= \left(\frac{\partial u}{\partial x}\right) v(y) ;$$

Similarly $\frac{\partial \psi}{\partial y} = u(x)\frac{\partial v}{\partial y}$. Subs. into our eqn:

$$\frac{\partial u(x)}{\partial x} v(y) + f(x)\, g(y)\, u(x) \frac{\partial v(y)}{\partial y} = 0 .$$

so,

$$\left(\frac{\partial u}{\partial x}\right) \frac{1}{f u} + \frac{g(y)}{v(y)} \frac{\partial v(y)}{\partial y} = 0$$

Now since u and f are <u>only</u> function of x and g, v are only functions of y, above can only be satisfied if

$$\boxed{\frac{\partial u}{\partial x} \frac{1}{f(x) u(x)} = \text{constant} = c}$$

$$\boxed{\frac{g(y)}{v(y)} \frac{\partial v(y)}{\partial y} = -c .}$$

Solution: $\dfrac{du}{dx} = c\, f(x)\, u(x)$.

$$\frac{dV}{dy} = -c\,\frac{V(y)}{g(y)}$$

$\therefore$ $\displaystyle\int \frac{du}{u} = c\int f(x)\,dx$. $\Rightarrow u(x) = c_1 e^{c\int f(x)dx}$

$\displaystyle\int \frac{dV}{V} = -c\int \frac{1}{g(y)}\,dy$. $V(y) = c_2 e^{-c\int dy/g(y)}$

So $$\boxed{\psi(x,y) = c_3\, e^{c\int f(x)dx}\, e^{-c\int dy/g(y)}}$$

$C_3 = \text{const}$

Ex: Solve $\dfrac{\partial\psi}{\partial x} + xy\dfrac{\partial\psi}{\partial y} = 0$.

So $f(x) = x$, $g(y) = y$

$$\psi(x,y) = (\text{const})\, e^{c\int x\,dx}\, e^{-c\int \frac{dy}{y}}$$

$$= \text{const}\ e^{cx^2/2} / y^c$$

Some examples of 1st order PDE in Physics.

Magnetic fields / Magnetic Potential.

Since there are no magnetic monopoles (seemingly) in nature, the physical magnetic field $\bar{B}$ is divergence-free, i.e.

$$\bar{\nabla}\cdot\bar{B} = \partial_x B_x + \partial_y B_y + \partial_z B_z = 0.$$

This equation means that we can always write

$$\bar{B} = \bar{\nabla}\times\bar{A} \quad , \quad \bar{A} = \text{magnetic potential}.$$

Example: 1

Lets take $\bar{B}$ to be constant and pointing along z-axis

$$\bar{B} = \hat{z}B \; ; \quad B = \text{constant}.$$

So, recalling that in components:-

$$\bar{\nabla}\times\bar{A} = (\partial_y A_z - \partial_z A_y)\hat{x} + (-\partial_x A_z + \partial_z A_x)\hat{y} + (\partial_x A_y - \partial_y A_x)\hat{z}$$

then we have to solve the 3 equations:-

$$\partial_y A_z - \partial_z A_y = 0$$

$$-\partial_x A_z + \partial_z A_x = 0$$

$$\partial_x A_y - \partial_y A_{\cancel{z}} = B$$

Let's assume $A_z = 0$;

1. $\partial_z A_y = 0$.
2. $\partial_z A_x = 0$
3. $\partial_x A_y - \partial_y A_x = B$.

$$\left.\begin{aligned} (1) &\Rightarrow \quad A_y = A_y(x,y) \\ (2) &\Rightarrow \quad A_x = A_x(x,y) \end{aligned}\right\} \text{indep. of } z.$$

put back into (3)

$$\partial_x A_y(x,y) - \partial_y A_x(x,y) = \underset{\uparrow \text{ constant}}{B}.$$

~~keeping y held fixed,~~

$$A_y = \alpha x B$$

$$\cancel{A_y =} \quad A_x = (\alpha - 1) y B.$$

solves above.

$$\text{so} \quad \bar{A} = (\alpha-1) y B \,\hat{x} + \alpha x B \,\hat{y}$$

solves $\underline{\bar{\nabla} \times \vec{A} = B\hat{z}}$

Ex 2: In mechanics, $\bar{F} = -\bar{\nabla} V$

where $V =$ potential.

Given $\bar{F} = (x^2+y^2+z^2)(x\hat{x} + y\hat{y} + z\hat{z})$.

Calculate $V(x,y,z)$.

In components,

$$-\frac{\partial V}{\partial x} = F_x = (x^2+y^2+z^2)x \quad (1)$$

$$-\frac{\partial V}{\partial y} = F_y = (x^2+y^2+z^2)y \quad (2)$$

$$-\frac{\partial V}{\partial z} = F_z = (x^2+y^2+z^2)z \quad (3)$$

Solve (1), keeping y,z fixed:

$$\frac{\partial V}{\partial x} = -(x^2+y^2+z^2)x \Big|_{y,z=\text{constant}}$$

$$V(x,y,z) = -\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) + \overbrace{f(y,z)}^{\text{indep of } x}$$

Subs into (2)

$$-\frac{\partial V}{\partial y} = -\frac{\partial}{\partial y}\left(-\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) + f(y,z)\right) = (x^2+y^2+z^2)y$$

$$+x^2y - \frac{\partial f}{\partial y} = x^2y + y^3 + z^2y$$

$$-\frac{\partial f}{\partial y} = y^3 + z^2y \; ; \Rightarrow f(y,z) = -\frac{y^4}{4} - z^2\frac{y^2}{2} + g(z)$$

To determine $g(z)$, subs back into 3.

$$-\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(-\frac{x^4}{4} - \frac{x^2}{2}(y^2+z^2) - \frac{y^4}{4} - \frac{z^2y^2}{2} + g(z)\right)$$

$$= (x^2+y^2+z^2)\,z.$$

$$+\cancel{x^2 z} + \cancel{y^2 z} - \frac{\partial g}{\partial z} = \cancel{x^2 z} + \cancel{y^2 z} + z^3$$

$$\frac{\partial g}{\partial z} = -z^3 \quad ; \quad g(z) = -\frac{z^4}{4}.$$

So finally,

$$V(x,y,z) = -\frac{x^4}{4} - \frac{y^4}{4} - \frac{z^4}{4} - \frac{x^2}{2}(y^2+z^2) - \frac{z^2y^2}{2}$$

$$= -\frac{1}{4}\left(x^4+y^4+z^4 + 2x^2y^2 + 2x^2z^2 + 2y^2z^2\right)$$

$$\underline{V(x,y,z) = -\frac{1}{4}\left(x^2+y^2+z^2\right)^2} = \text{potential.}$$