School of Mathematical Sciences
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MTH5112 Linear Algebra I
TEST

Date: 10 November 2011 *Time:* 3.00–3.40pm

Arts 2 Lecture Theatre	Surnames A to M
Mason Lecture Theatre	Surnames N to Z

Complete the following information:

Name	**SOLUTION**

The duration of the test is **40 minutes**. Answer **all** questions **in the spaces provided**. Write the final answer clearly. Calculators are **not** allowed.

Total Marks	

Nothing on this page will be marked!

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1. Let A be an $m \times n$ real matrix. Give a definition of the *null space*, $N(A)$, of A.

Let

$$A = \begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & -1 & 4 & -1 \\ -2 & -4 & 7 & -1 \end{pmatrix}.$$

Determine the null space $N(A)$.

The null space $N(A)$ is given by

$$N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \in \mathbb{R}^n : A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \right\}.$$

To determine

$$N(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} : \begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & -1 & 4 & -1 \\ -2 & -4 & 7 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\},$$

we reduce A to echelon form

$$\begin{pmatrix} 1 & 2 & -3 & 1 \\ -1 & -1 & 4 & -1 \\ -2 & -4 & 7 & -1 \end{pmatrix} \xrightarrow[2R_1+R_3]{R_1+R_2} \begin{pmatrix} 1 & 2 & -3 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

and set free variable $x_4 = t$. Then $x_3 + x_4 = 0 \Rightarrow x_3 = -t$; $x_2 + x_3 = 0 \Rightarrow x_2 = t$; $x_1 + 2x_2 - 3x_3 + x_4 = 0 \Rightarrow x_1 = -6t$.

Hence $N(A) = \left\{ \begin{pmatrix} -6t \\ t \\ -t \\ t \end{pmatrix} : t \in \mathbb{R} \right\} = \left\{ t \begin{pmatrix} -6 \\ 1 \\ -1 \\ 1 \end{pmatrix} : t \in \mathbb{R} \right\}.$

2. Let

$$A = \begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix}.$$

Give a reason to show that A is invertible.

Further, find the $(1,3)$-entry of the inverse A^{-1}.

We have

$$\det\begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 2 & 2 & 3 \end{pmatrix} \underset{-R_1+R_3}{=} \det\begin{pmatrix} 1 & 4 & 3 \\ -1 & -2 & 0 \\ 1 & -2 & 0 \end{pmatrix}$$

$$= 3\begin{vmatrix} -1 & -2 \\ 1 & -2 \end{vmatrix} = 3\Big((-1)(-2) - (-2)(1)\Big) = 12 \neq 0$$

Hence A is invertible.

Let $\operatorname{adj}(A) = (C_{ij})^T$ be the adjugate of A, where

$C_{ij} = (-1)^{i+j}\det A_{ij}$, A_{ij} is the (i,j)-minor of A.

Then $$A^{-1} = \frac{1}{\det(A)}\operatorname{adj}(A) = \frac{1}{12}(C_{ji})$$

The $(1,3)$-entry of $\operatorname{adj}(A)$ is $C_{31} = (-1)^{3+1}\det A_{31}$

$$= \begin{vmatrix} 4 & 3 \\ -2 & 0 \end{vmatrix} = 6.$$

Hence the $(1,3)$-entry of A^{-1} is

$$\frac{1}{\det A}C_{31} = \frac{6}{12} = \frac{1}{2}.$$

3. Determine, **with a reason**, whether the following vectors are linearly independent in the vector space $\mathbb{R}^4$:

$$\begin{pmatrix}2\\0\\1\\1\end{pmatrix},\ \begin{pmatrix}0\\1\\6\\1\end{pmatrix},\ \begin{pmatrix}0\\0\\2\\-2\end{pmatrix},\ \begin{pmatrix}1\\0\\0\\3\end{pmatrix}.$$

Do they form a basis of $\mathbb{R}^4$?

We compute the determinant

$$\begin{vmatrix}2&0&0&1\\0&1&0&0\\1&6&2&0\\1&1&-2&3\end{vmatrix} = 2\begin{vmatrix}1&0&0\\6&2&0\\1&-2&3\end{vmatrix} + (-1)\begin{vmatrix}0&1&0\\1&6&2\\1&1&-2\end{vmatrix}$$

$$= 2\begin{vmatrix}2&0\\-2&3\end{vmatrix} - (-1)\begin{vmatrix}1&2\\1&-2\end{vmatrix} = 2(6-0)+(-2-2)$$

$$= 12-4 = 8 \neq 0.$$

Since the determinant is nonzero, the given vectors are linearly independent.

Yes, they form a basis since $\dim \mathbb{R}^4 = 4$ implies that they span $\mathbb{R}^4$.

4. Let V be a real vector space and let $x, y, z \in V$. Give a definition of the linear span

$$\mathrm{Span}\,(x, y, z)$$

of the vectors x, y and z.

Let x, y, z be linearly independent. Show that each vector v in Span (x, y, z) is a **unique** linear combination of x, y and z.

$\mathrm{span}(x,y,z) = \{\alpha x + \beta y + \gamma z : \alpha, \beta, \gamma \in \mathbb{R}\}$.

Let $v \in \mathrm{span}(x,y,z)$ and let v have two representations

$$v = \alpha x + \beta y + \gamma z = \alpha' x + \beta' y + \gamma' z.$$

Then $(\alpha - \alpha')x + (\beta - \beta')y + (\gamma - \gamma')z = 0$.

Since x, y, z are linearly independent, we must have $\alpha - \alpha' = \beta - \beta' = \gamma - \gamma' = 0$. Hence v is a unique linear combination of x, y, z.

Let $\mathbb{R}^{4\times 4}$ be the real vector space of 4×4 real matrices. Determine, **with a reason**, if the subset

$$W = \{A \in \mathbb{R}^{4\times 4} : \det A = 0\}$$

is a subspace of $\mathbb{R}^{4\times 4}$.

(We have indeed $0 \in W$) let

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $\det A = \det B = 0$. Hence $A, B \in W$.

But $\det(A+B) = \det \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = 1 \neq 0$

and $A + B \notin W$. Hence W is not a subspace of $\mathbb{R}^{4\times 4}$.

END