ASTM001/MAS423 Solar System
Tides

Craig B. Agnor

Queen Mary, University of London

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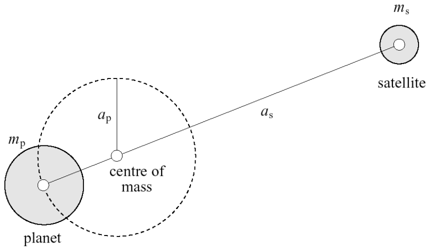
Introduction

- Real bodies in the solar system are not point masses.
- The finite dimensions of a body mean that, in the presence of another mass, there will be a *gravitational gradient* across the body. This gives rise to a *tidal bulge*.
- Measurements of the amplitude of the bulge provide information about the internal strucure of the body. Rotational distortion can give similar information.
- The response of a satellite to the tidal bulge it raises on a planet can lead to orbital evolution of the satellite.

The Tidal Bulge

The magnitude of the mean force between a planet and a satellite is

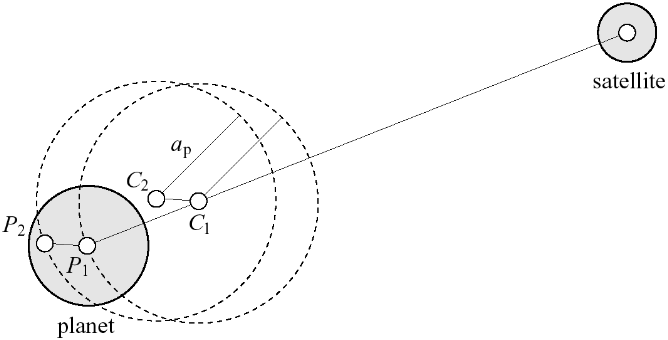
$$\langle F \rangle = G\frac{m_p m_S}{r^2}$$
$$\frac{a_s}{a_p} = \frac{m_p}{m_s}$$
$$a = a_p + a_s$$



Here they both orbit about the mutual center of mass in circular orbits.

The Tidal Bulge

Consider the paths of particles inside the planet, ignoring the rotation of the planet.



All particles within th eplanet experience the same centrifugal force (in magnitude and direction) but a different gravitational force, **F**, because of their different distances from the satellite.

The Tidal Bulge

Hence:

$$\langle \mathbf{F} \rangle = \text{centrifugal force} \neq \mathbf{F}$$

The tide generating force is then:

$$\mathbf{F}_{Tidal} = \mathbf{F} - \langle \mathbf{F} \rangle$$

Rotational forces also deform a body, but if tidal and rotational deformations are small then they can be treated separately and added.

We are actually more concerned with the gravitational potential that gives rise to the tidal bulge. This is because if a body is in hydrostatic equilibrium then its surface is equipotential.

The Tidal Bulge

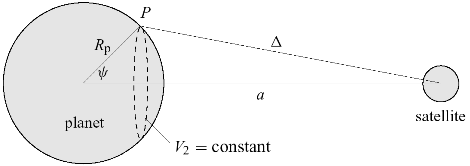
Consider the potential V, at some point, P, on the surface of the planet. Then we have

$$V = -G\frac{m_s}{\Delta}; \qquad \frac{\mathbf{F}}{m} = -\nabla V$$

$$\nabla \equiv \left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial \psi}, \frac{1}{r\sin\psi}\frac{\partial}{\partial \phi}\right)$$

and

$$\frac{\mathbf{F}}{m} = -\frac{\partial}{\partial r}\left(\frac{GM}{r}\right)\hat{\mathbf{r}} = -\frac{GM}{r^2}\hat{\mathbf{r}} = -\frac{GM}{r^3}\mathbf{r}$$



$$\Delta^2 = a^2 + R_p^2 - 2aR_p\cos\psi$$

Tidal Bulge

We expanded Δ in terms of (R_p/a) and $cos\psi$ and find the tide raising potential can be expressed as

$$V_3(\psi) = -G\frac{m_s}{a^3}R_p^2 P_2(\cos\psi)$$

where $P_2(x) = \frac{1}{2}(3x^2 - 1)$ is the Legendre polynomial of degree 2. Since,

$$\frac{\mathbf{F}_{tidal}}{m_p} = \frac{\mathbf{F}}{m_p} - \frac{\langle \mathbf{F} \rangle}{m_p} = -\nabla V - \frac{\langle \mathbf{F} \rangle}{m_p} \approx -\nabla V_3(\psi)$$

this is the tide raising part of the potential.

The Tidal Bulge

This potential can also be written as:

$$V_3(\psi) = -\zeta g P_2(\cos\psi)$$

where

$$\zeta = \frac{m_s}{m_p}\left(\frac{R_p}{a}\right)^3 R_p \qquad g = \frac{Gm_p}{R_p^2} \tag{1}$$

The Tidal Bulge - Amplitude

- Here the quantity, $\zeta P_2(\cos\psi)$ is the amplitude of the equilibrium tide.
- A plot of $P_2(\cos\psi) = \frac{1}{2}(3\cos^2\psi - 1)$ as a function of ψ, shows two maxima and two minima per cycle. This is why there are two tides per day.
- For the tide raised on the Earth by the Moon, $\zeta = 0.36$m
- For the tide raised on the Earth by the Sun, $\zeta = 0.16$m
- These interfere constructively/destructively depending on orientation and phase of the Sun and Moon.
- And they have important other implications as well,. . .

Tidal Friction

Tidal oscillations always generate friction and this results in energy loss and a phase shift in the tidal response of the planet.
The specific or tidal dissipation function Q is a measure of how much dissipation is produced:

$$Q = \frac{2\pi E_o}{\Delta E}$$

Here ΔE is the energy lost over one cycle and E_o is the peak energy stored during the cycle.
The tidal bulge will be carried ahead or lag behind the satellite by an amount,ϵ. Note from earlier that this is related to the phase shift, δ, and Q by

$$\sin 2\epsilon = \sin \delta = \frac{1}{Q_p}$$

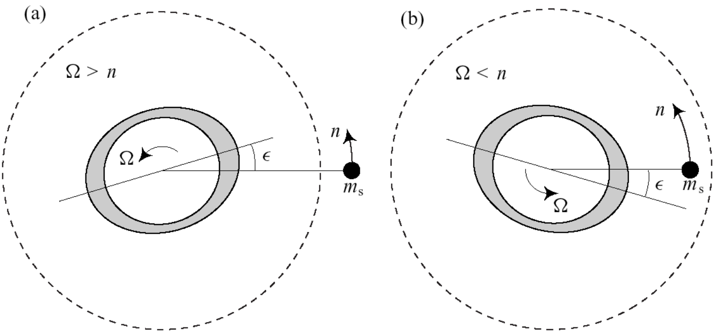
Tidal Friction - Synchronous Orbit

- Lagging ahead or behind depends on the satellites orbit relative to the synchronous orbit.
- A synchronous orbit is one where the mean motion (or angular velocity) (n) is equal to that of the primary's rotational angular velocity Ω_p.
- The radius of the synchronous orbit can then be found from Kepler's 3rd Law.

$$a_{synch} = \left(\frac{Gm_p}{\Omega_p^2} \right)^{1/3}$$

Tidal Torque

The response of the satellite to the tidal bulge it has raised depends on whether the satellite is (a) outside synchronous orbit or (b) inside synchronous orbit. The resulting asymmetry exerts a torque on the satellite.



Tidal Torques

The torque, $\mathbf{\Gamma}$, is determined by the cross product of the radius with the force exerted on the satellite by the external potential of the deformed planet. Hence

$$\mathbf{\Gamma} = \mathbf{r} \times \mathbf{F} \text{ where } \mathbf{F} = -m_s \nabla V_{ext}$$

Only the component of the force perpendicular to the line joining the planet-satellite centers contributes to the torque.

$$F_\psi = -\left(\frac{m_s}{r}\right)\left(\frac{\partial V_{ext}}{\partial \psi}\right)$$

Torques and Energy Dissipation

For linear motion, the rate of change of energy (Power) can be computed with

$$P = \mathbf{F} \cdot \mathbf{v}$$

The rotational analog to this is

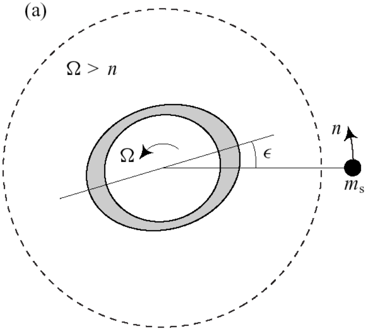
$$P = \mathbf{\Gamma} \cdot \omega$$

where ω is some angular velocity.
In the present problem we have both rotational and orbital angular velocities.

Torque & Power: Above Synchronous

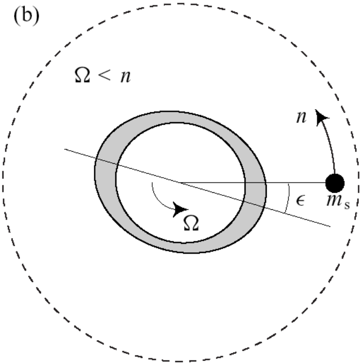
- Bulge leads ahead satellite and a torque is applied
- Work increase satellite's orbital energy at a rate $\dot{E}_s = \Gamma n$.
- An equal an opposite torque is applied to the planet
- Work decreases planet's rotational energy at a rate of $\dot{E}_p = -\Gamma\Omega_p$
- These rates are *not* equal.
- Mechanical energy is lost at a rate of

$$\dot{E} = \dot{E}_p + \dot{E}_s = -\Gamma(\Omega_p - n) < 0$$



Torque & Power: Below Synchronous

- Bulge lags behind satellite and a torque is applied
- Work decreases satellite's orbital energy at a rate $\dot{E}_s = -\Gamma n$.
- An equal an opposite torque is applied to the planet
- Work increases planet's rotational energy at a rate of $\dot{E}_p = \Gamma \Omega_p$
- Mechanical energy is lost at a rate of $\dot{E} = \dot{E}_p + \dot{E}_s = \Gamma(\Omega_p - n) < 0$



Where does the energy go?
What about angular momentum? Is it being lost?

Tidal Energy Dissipation

In both cases the energy is dissipated as heat in the planet. The rate of energy dissipation determines the rate of orbital evolution. The total mechanical energy of the system is the sum of rotational and orbital contributions. The rate of chane is

$$\frac{d}{dt}E = \frac{d}{dt}\left(\frac{1}{2}I\Omega_p^2 - G\frac{m_p m_s}{2a}\right) = I\Omega_p\dot{\Omega}_p + G\frac{m_p m_s}{2a^2}\dot{a}$$

where $I = 2/5MR^2$ is the moment of inertia of a sphere. Kepler's 3rd Law, $G(m_p + m_s) = n^2a^3$ allows us to write

$$\dot{E} = I\Omega_p\dot{\Omega}_p + \frac{1}{2}\frac{m_p m_s}{(m_p + m_s)}n^2a\dot{a}$$

and this must be negative as mechanical energy is being dissipated as heat.

Tidal Torques and Angular Momentum

However, the total angular momentum of the system is conserved.

$$L = I\Omega_p + \frac{m_p m_s}{(m_p + m_s)} a^2 n$$

$\dot{L} = 0$ and hence

$$I\dot{\Omega}_p = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a}$$

This expression links orbital evolution of the satellite to the spin. We can write subsequent expressions either in terms of $\dot{\Omega}_p$ or $\dot{a}$. If we can find one, we also get the other.

A Common Substitution

Above have used the expression below to get a simple form.

$$\frac{d}{dt}(a^2 n) = 2a\frac{da}{dt}n + a^2\frac{dn}{dt}$$

and

$$\frac{d}{dt}(n) = \frac{d}{dt}\left(\frac{\sqrt{G(m_p + m_s)}}{a^{3/2}}\right) = -\frac{3}{2}\frac{n}{a}\frac{da}{dt}$$

so

$$\frac{d}{dt}\left(a^2 n\right) = \left(2an - \frac{3}{2}an\right)\dot{a} = \frac{1}{2}an\dot{a}$$

Tidal Dissipation

The rate of change of mechanical energy can be written:

$$\dot{E} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} (\Omega_p - n) \quad (2)$$

Since this must be negative we have:

$$\text{sgn}(\dot{a}) = \text{sgn}(\dot{\Omega}_p) = \text{sgn}(\Omega_p - n) \quad (3)$$

Therefore:

- Satellite in prograde orbits beyond the synchronous orbit will move outward.
- Satellites in prograde orbits inside the synchronous orbit will move inwards.

  (Also, satellites in retrograde orbits will move inward)

. . . *yeah, yeah, but how fast?*

The Potential the Satellite sees

The tidal potential due to the satellite that distorts the primary is:

$$V_3(\psi) = -\zeta g P_2(\cos\psi)$$

The amplitude of the tide that arises in the primary is

$$C\epsilon_2 = \frac{(5/2)\zeta}{1+\tilde{\mu}}$$

This is the amplitude (height) of the tide and it depends on the physical response of the body to tidal distortion. Here C is the mean radius and Here $\tilde{\mu}$ is the 'effective' rigidity.

$$\tilde{\mu} = \frac{19\mu}{2\rho g R}$$

and μ is the physical rigidity.

- $\mu = 5 \times 10^{10}$ N m^{-2} for rock
- $\mu = 4 \times 10^{9}$ N m^{-2} for ice

The Love Numbers

In general, the tidal amplitude in the primary is

$$C\epsilon_2 = \frac{(5/2)\zeta}{1+\tilde{\mu}} = h_2\zeta$$

Where h_2 and k_2 are *the Love numbers*

$$h_2 = \frac{(5/2)}{1+\tilde{\mu}} \qquad k_2 = \frac{(3/2)}{1+\tilde{\mu}}$$

These parameterize the rigidity of the body in response to gravity.
$k_2 = 0 \rightarrow 3/2$
$h_2 = 0 \rightarrow 5/2$

What the Satellite Sees . . .

. . . is a primary that *it* deformed. This part of the potential associated with the deformation.

$$V_{nc,ext} = -k_2 \zeta g \left(\frac{C}{r}\right)^3 P_2(\cos\psi)$$

recall that

$$\zeta = \frac{m_s}{m_p}\left(\frac{C}{r}\right)^3 R_p$$

Using the mean radius $R_p = C$ and $r = a$, the potential the satellite sees due to its deformation of the primary is

$$V_{nc,ext} = -k_2 \frac{Gm_s}{C}\left(\frac{C}{r}\right)^6 P_2(\cos\psi)$$

The Tidal Torque

Now we can compute the magnitude of the torque and resulting orbital evolution.

$$\Gamma = |\Gamma| = m_s \frac{\partial V_{nc,ext}}{\partial \psi}$$

$$\frac{\partial P_2(\cos\psi)}{\partial \psi} = -\frac{3}{2}\sin 2\psi$$

and the resulting torque has magnitude

$$\Gamma = \frac{3}{2} k_2 \frac{G m_s^2}{a^6} C^5 \sin 2\epsilon$$

note dependence on k_2, m_s, and a. This can then be substituted into our expression for $\dot{E}$, $\dot{\Omega}_p$ and $\dot{a}$.

The rates of change

We know that:

$$\dot{E} = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} (\Omega_p - n)$$

and

$$\dot{E} = -\Gamma(\Omega_p - n)$$

we get

$$\Gamma = -\frac{1}{2}\frac{m_p m_s}{(m_p + m_s)} n a \dot{a} = \frac{3}{2}\frac{k_2}{Q_p}\frac{G m_s^2}{a^6} C^5$$

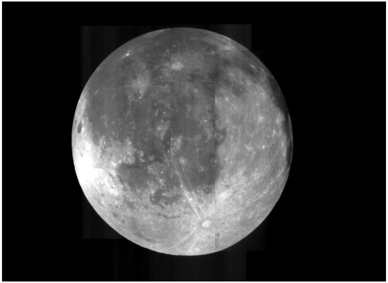
and

$$\dot{a} = \mathrm{sgn}(\Omega_p - n)\frac{3k_2}{Q_p}\frac{m_s}{m_p}\left(\frac{C_p}{a}\right)^5 n a$$

(you should confirm the algebra and substitutions here, e.g. Kepler's 3rd $n^2 = G(m_p + m_s)/a^3$)

Observable Consequences #1

- Apollo astronauts placed reflectors on the lunar surface. These are used to very accurately track the Moon's movements.
- For the Moon, $\dot{a} \approx +10^{-9}$ m/s or about 3cm per year.
- As the Moon moves away, the Earth's spin slows down. Shorter days are recorded in the fossil record.



Tidal Orbital Evolution

The expression for $\dot{a}$ can be easily integrated to give

$$\frac{2}{13} a^{13/2} \left[1 - \left(\frac{a_o}{a} \right)^{13/2} \right] = \frac{3k_{2p}}{Q_p} \left(\frac{G}{m_p} \right)^{1/2} C_p^5 m_s \Delta t$$

where a_o is the initial semimajor axis and a is its value after time Δt.

If we ignore the second term on the LHS then:

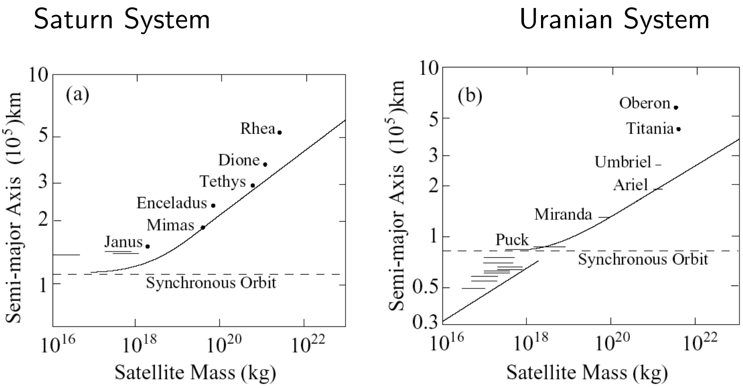
$$\frac{2}{13} a^{13/2} = \frac{3k_{2p}}{Q_p} \left(\frac{G}{m_p} \right)^{1/2} C_p^5 m_s \Delta t$$

or

$$\log a = \frac{2}{13} \log m_s + \text{constant}$$

for tidally evolved systems.

Observable Consequences #2



- In both systems there seems to be evidence for tidal evolution, with the more massive satellites lying further away from the planet.
- We can use current a and $a_o = a_{synch}$ to estimate lower limit on Q_p.

‘Planet’ or Primary Tides - Summary

- Tidal bulge raised in primary dissipate energy, heating the primary.
- Tidal bulge and phase lag due to friction allow exchange of angular momentum between orbit of secondary (satellite) and spin of primary (planet).
- Rate of exchange is controlled by masses, separations, as well as the rigidity of the primary and the frictional properties of the primary (which are poorly known, and not likely constant).
- We can use observed locations of satellites near synchronous orbit and assume they formed *right at* the synchronous orbit, to estimate Q_p of a planet or primary.

Despinning

From angular momentum conservation we can obtain despinning rates. If $I_p = \alpha_p m_p C_p^2$ (where $\alpha_p \leq 2/5$)

$$\dot{\Omega}_p = -\,\mathrm{sgn}(\Omega_p - n)\frac{3k_{2p}}{2\alpha_p Q_p}\frac{m_s^2}{m_p(m_p + m_s)}\left(\frac{C_p}{a}\right)^3 n^2 \qquad (4)$$

The planet also raises a tide in the satellite; facilitating angular momentum exchange.

We can use precisely the same model to estimate the torque, and spin evolution of the secondary.

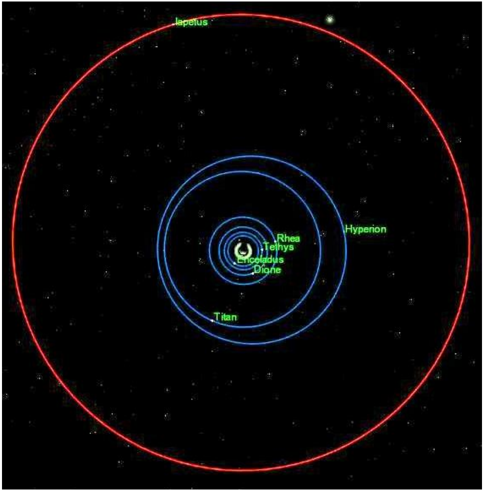
$$\dot{\Omega}_s = -\,\mathrm{sgn}(\Omega_s - n)\frac{3k_{2s}}{2\alpha_s Q_s}\frac{m_p^2}{m_s(m_p + m_s)}\left(\frac{C_s}{a}\right)^3 n^2$$

where s denotes the satellite. Note the symmetry and mass ratio dependence.

Generally, $m_s \ll m_p$, and spin-down of the satellite is *much* faster.

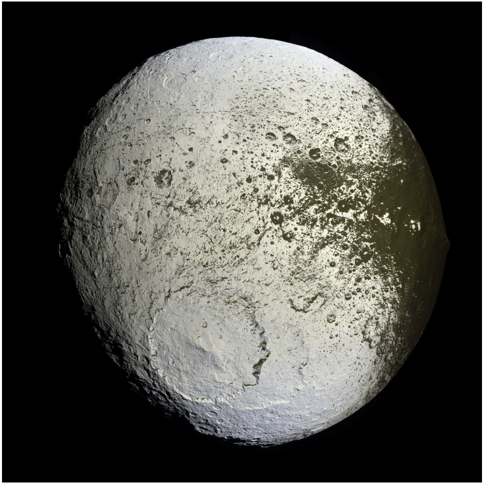
Iapetus - orbital characteristics

- Semimajor axis $a = 3.56 \times 10^6 \text{km} = 59 R_S$
- $e = 0.0286125$
- Prograde and inclined:
  17.28° (to the ecliptic)
  15.47° (to Saturn's equator)
  7.52° (to local Laplace plane)

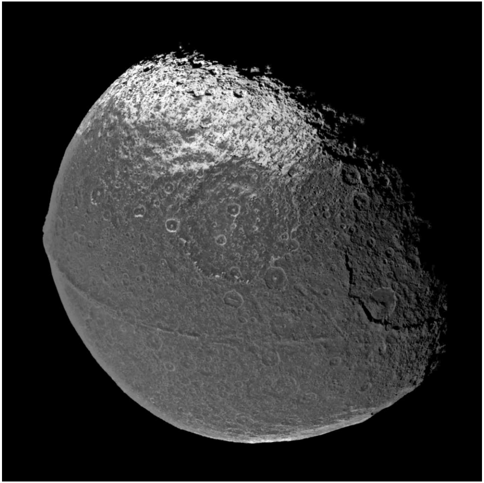


Iapetus - physical characteristics

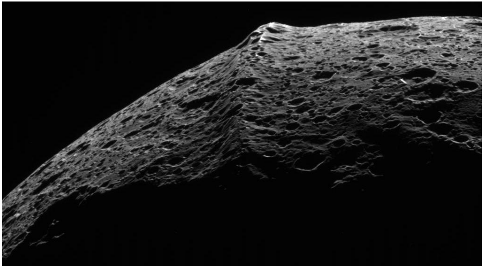
- A *'mid-sized'* icy satellite of Saturn
- Mean radius $R = 735$ km
- Triaxial ellipsoid: $\approx 747 \times 749 \times 712$ km
- Mean density $\rho = 1083.0$ kg m^{-3}, (mostly ice)
- Heavily cratered ancient surface
- Slow *synchronous* rotation (79.3215 days)
- Has gross dark/light global shading like a *frosted mini-wheat*.



Iapetus - Cassini's 2004 flyby



- Revealed Iapetus' equatorial ridge
- Ridge height varies 10-20 km
- Sections over 1300 km
- Ridge is cratered $\rightarrow$old
- Ridge only appears in the dark colored area (coverage is yet not complete)
- *What produced the ridge?*



What if Iapetus had its own Rings/Satellites!

- Iapetus has a large Hill sphere relative to Saturn.
- It could sustain a ring system shortly after the satellite's formation.
- As Iapetus despun the ring system collapsed to the satellite surface $\rightarrow$ the ridge is a pile-up of the debris of the ring system material.
- However, ...
  - The ridge appears quite solid rather than made of rubble.
  - Recent images show the ridge to have faults running along it.

What if Iapetus formed molten and spinning rapidly?

- A rotating fluid Iapetus would be rotationally flattened.
- As Iapetus cooled it formed a thick outer ice shell (which has some strength/rigidity).
- Tides from Saturn resulted in despinning of Iapetus
  - Oblate triaxial ellipsoid is no longer supported by rotation.
  - Stress causes shell to buckle at the equator as Iapetus is despun.
- Requires that despinning takes longer than cooling and thickening of ice shell (both things we can easily calculate/estimate).

Neither model explains why the ridge only appears in Iapetus' dark colored region. . . . Stay Tuned.