Distance Regularity in Direct-Product Graphs

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(Received March 1999; accepted April 1999)

Abstract—The direct product (also called Kronecker product, tensor product, and cardinal product) \(G \times H\) of distance-regular graphs is investigated. It is demonstrated that the product is distance-regular only when \(G\) and \(H\) are very restricted distance-regular graphs. @ 1999 Elsevier Science Ltd. All rights reserved.

Keywords—Distance-regular graphs, Direct product.

1. INTRODUCTION

Every connected graph is representable by means of a level diagram (cf. [1]) as follows. Choose a vertex \(u\), and let it be the sole resident of level zero. The vertices on level \(i\) are precisely those whose distance from \(u\) is \(i\). Now add edges of the graph and note that the edges occur only between vertices of adjacent levels and among vertices of the same level. Distance regularity is definable in terms of a level diagram. Let \(d\) be the diameter of a given graph \(G\), and let \(v\) be a vertex of \(G\) on level \(i\). Further, let

\[
\begin{align*}
  a_i &= \text{number of vertices on level } i \text{ adjacent to } v, \quad i = 1, \ldots, d, \\
  b_i &= \text{number of vertices on level } i+1 \text{ adjacent to } v, \quad i = 0, \ldots, d-1, \\
  c_i &= \text{number of vertices on level } i-1 \text{ adjacent to } v, \quad i = 1, \ldots, d.
\end{align*}
\]

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We got to know about the unique characteristic of the (3, 12)-cage from C. Godsil, and received much-needed encouragement from P. Weichsel. We are also thankful to the referee, whose comments on the earlier draft led to an improvement in the paper.
G is said to be *distance-regular* if it is connected and the numbers $a_i$, $b_i$, and $c_i$ depend only on $i$ and not on the choice of the level diagram or on the choice of $v$. A distance-regular graph is necessarily regular. On the other hand, every regular graph need not be distance-regular. Cycles, complete graphs, and hypercubes are some of the familiar graphs easily seen to be distance-regular. Brouwer et al. [2] list several characteristics and applications of this class of graphs.

What distance-regular graphs survive stress of the direct product? It turns out that most such graphs are of low diameter. In particular, if $G$ and $H$ are distance-regular graphs with $d(G) > 2$ and $d(H) > 2$, then $G \times H$ is not distance-regular. (Here $d(G)$ denotes the diameter of $G$.) This is to be contrasted with Weichsel's result with respect to the Cartesian product [3].

Let $G$ be a distance-regular graph, and let $\Delta$ be the degree of $G$. It is easy to see that $b_0 = \Delta$ and $c_1 = 1$. Further, if $G$ is bipartite, then $a_i = 0$ for all $i$ and $c_d = \Delta$. The pair of sequences $[(b_0, \ldots, b_{d-1}); (c_1, \ldots, c_d)]$ is called the *intersection array*. It contains all the essential information about the graph but falls short of uniquely determining the graph.

By a graph is meant a finite, simple, and undirected graph. Unless indicated otherwise, graphs are also connected and have at least two vertices. Let $G = (V, E)$ and $H = (W, F)$ be graphs. The *direct product* $G \times H$ of $G$ and $H$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{\{(u,x),(v,y)\} : \{u,v\} \in E \text{ and } \{x,y\} \in F\}$. This product is variously known as Kronecker product, tensor product, and cardinal product. Certain relevant characteristics are as follows:

(i) $G \times H$ is bipartite iff $G$ or $H$ is bipartite;
(ii) $G \times H$ is connected iff $G$ or $H$ is nonbipartite;
(iii) if $G$ and $H$ are both bipartite, then $G \times H$ consists of two connected components; and
(iv) if $G = (V_0 \cup V_1, E)$ is a bipartite graph equipped with an automorphism that swaps the two colors, then for every bipartite graph $H$, the two components of $G \times H$ are isomorphic to each other [4].

Note that the $(3, 12)$-cage is bipartite and distance-regular, yet it does not admit of an automorphism swapping the two colors [2].

### 2. RESULTS

Our important result (that appears in Corollary 2.3 below) is that if $d(G) > 2$ and $d(H) > 2$, then $G \times H$ is not distance-regular. Theorem 2.4 is a characterization for distance-regularity of $G \times H$ where $d(G) = d(H) = 2$. Finally, we deal with the case when one or each of $G$ and $H$ is a complete graph and state certain results relating to $G \times K_2$.

**Theorem 2.1.** Let $G$ and $H$ be distance-regular graphs such that $G$ or $H$ is nonbipartite, and $d(G) \geq 2$, $d(H) \geq 2$. For $2 \leq k \leq \min\{d(G), d(H)\}$, if $c_k(G) \neq \Delta(G)$ or $c_k(H) \neq \Delta(H)$, then $G \times H$ is not distance-regular.

**Proof.** Let $G$, $H$, and $k$ be as stated, and note that $G \times H$ is connected. For a vertex $u$ of $G$, consider the level diagram of $G$ with $u$ at level zero. Similarly, for a vertex $v$ of $H$, consider the level diagram of $H$ with $v$ at level zero. We construct the level diagram of $G \times H$ with vertex $(u,v)$ at level zero.

We first claim that for each $k$, there exist vertices $(x_1, y_1)$, $(x_2, y_2)$, and $(x_3, y_3)$ at level $k$ of $G \times H$ such that

(a) $\text{dist}_G(u, x_1) = k = \text{dist}_H(v, y_1)$,
(b) $\text{dist}_G(u, x_2) = k > \text{dist}_H(v, y_2)$, and
(c) $\text{dist}_G(u, x_3) < k = \text{dist}_H(v, y_3)$, where $\text{dist}_G(u, x_i)$ and $\text{dist}_H(v, y_i)$ are of the same parity, $1 \leq i \leq 3$.

For (a), note that $G$ is distance-regular, and $2 \leq k \leq \min\{d(G), d(H)\}$, hence there exist vertices $x_1$ and $y_1$ in $G$ and $H$, respectively, such that $\text{dist}_G(u, x_1) = k = \text{dist}_H(v, y_1)$. Clearly, $\text{dist}_{G \times H}((u,v),(x_1,y_1)) = k.$
For (b), let \( u - u_1 - u_2 - \cdots - u_k \) be a shortest path of length \( k \) in \( G \), and let \( v_1 \) be a vertex adjacent to \( v \) in \( H \). If \( k \) is even, then \( (u, v) - (u_1, v_1) - (u_2, v) - \cdots - (u_{k-1}, v_1) - (u_k, v) \) is a shortest path of length \( k \) in \( G \times H \). On the other hand, if \( k \) is odd, then \( (u, v) - (u_1, v_1) - (u_2, v) - \cdots - (u_{k-1}, v) - (u_k, v_1) \) is a shortest path of length \( k \) in \( G \times H \). Letting \( x_2 = u_k \) and \( y_2 = v_1 \) if \( k \) is even, and letting \( x_2 = u_k \) and \( y_2 = v_1 \) if \( k \) is odd, we observe:

(i) \( \text{dist}_G(u, x_2) = k > \text{dist}_H(v, y_2) \), and
(ii) \( \text{dist}_G(u, x_2) \) and \( \text{dist}_H(v, y_2) \) are of the same parity.

Argument for (c) is analogous to the foregoing.

For each of the vertices of the type \((x_1, y_1), (x_2, y_2), \) and \((x_3, y_3)\) at level \( k \) of \( G \times H \) mentioned above, we compute \( c_k \).

- Vertex \((x_1, y_1)\) is adjacent to a vertex \((r_1, s_1)\) at level \( k - 1 \) iff \( \text{dist}_G(u, r_1) = k - 1 = \text{dist}_H(v, s_1) \), and \( x_1, r_1 \) (respectively, \( y_1, s_1 \)) are adjacent in \( G \) (respectively, \( H \)). The number of such vertices at level \( k - 1 \) of \( G \times H \) is exactly \( c_k(G) \cdot c_k(H) \).

- Vertex \((x_2, y_2)\) is adjacent to a vertex \((r_2, s_2)\) at level \( k - 1 \) iff \( \text{dist}_G(u, r_2) = k - 1 \), and \( x_2, r_2 \) are adjacent in \( G \). The number of such vertices at level \( k - 1 \) of \( G \times H \) is exactly \( c_k(G) \cdot \Delta(H) \).

- Vertex \((x_3, y_3)\) is adjacent to a vertex \((r_3, s_3)\) at level \( k - 1 \) iff \( \text{dist}_H(v, s_3) = k - 1 \), and \( y_3, s_3 \) are adjacent in \( H \). The number of such vertices at level \( k - 1 \) of \( G \times H \) is exactly \( \Delta(G) \cdot c_k(H) \).

If the graph \( G \times H \) is to be distance-regular, then \( c_k(G) \cdot c_k(H) = \Delta(G) \cdot \Delta(H) = \Delta(G) \cdot c_k(H) \). This implies that \( c_k(G) = \Delta(G) \) and \( c_k(H) = \Delta(H) \).

**Theorem 2.2.** If \( G \) and \( H \) are bipartite distance-regular graphs with \( d(G) \geq 2 \) and \( d(H) \geq 2 \), then each component of \( G \times H \) is distance-regular iff \( c_k(G) = \Delta(G) \) and \( c_k(H) = \Delta(H) \), where \( 2 \leq k \leq \min\{d(G), d(H)\} \).

**Proof.** Let \( G \) and \( H \) be as stated. \( G \times H \) consists of two (bipartite) components, where vertices \((u, v)\) and \((x, y)\) belong to the same component iff \( \text{dist}_G(u, x) \) and \( \text{dist}_H(v, y) \) are of the same parity. Observe that \( d(G \times H) = \max\{d(G), d(H)\} \), cf. \([5]\).

We construct a level diagram of one component of \( G \times H \) as in the proof of Theorem 2.1. With vertex \((u, v)\) at level zero, we distinguish among three types of vertices, namely, \((x_1, y_1), (x_2, y_2)\) and \((x_3, y_3)\), at level \( k \), where

(a) \( \text{dist}_G(u, x_1) = k = \text{dist}_H(v, y_1) \),
(b) \( \text{dist}_G(u, x_2) = k > \text{dist}_H(v, y_2) \), and
(c) \( \text{dist}_G(u, x_3) < k = \text{dist}_H(v, y_3) \).

Note that \( \text{dist}_G(u, x_i) \) and \( \text{dist}_H(v, y_i) \) are necessarily of the same parity, \( 1 \leq i \leq 3 \).

By an argument as in the proof of Theorem 2.1, if each component of \( G \times H \) is distance-regular, then \( c_k(G) = \Delta(G) \) and \( c_k(H) = \Delta(H) \), where \( 2 \leq k \leq \min\{d(G), d(H)\} \).

For the converse, assume that \( c_k(G) = \Delta(G) \) and \( c_k(H) = \Delta(H) \) for \( 2 \leq k \leq \min\{d(G), d(H)\} \). This means that each of \( G \) and \( H \) is of the form \( K_{n, n} \). In this case, each component of \( G \times H \) is again such.

**Corollary 2.3.** If \( G \) and \( H \) are distance-regular graphs with \( d(G) > 2 \) and \( d(H) > 2 \), then \( G \times H \) or a component of \( G \times H \) is not distance-regular.

**Proof.** Let \( G \) and \( H \) be as stated. Since \( b_2(G) > 0 \) for any distance-regular graph with diameter at least three, it is clear that \( c_2(G) < \Delta(G) \). By Theorems 2.1 and 2.2, \( G \times H \) or a connected component of \( G \times H \) is not distance-regular.

**Theorem 2.4.** Let \( G \) and \( H \) be distance-regular graphs with \( d(G) = d(H) = 2 \). Each component of \( G \times H \) is distance-regular iff both \( G \) and \( H \) are bipartite.

**Proof.** Let \( G \) and \( H \) be as stated. First, assume that \( G \) is nonbipartite, in which case either \( a_1(G) > 0 \) or \( a_2(G) > 0 \). If \( a_2(G) > 0 \), then \( c_2(G) < \Delta(G) \), and by Theorem 2.1, \( G \times H \) is not
distance-regular. Suppose that $a_1(G) > 0$. Let $u, x \in V(G)$, $v, y \in V(H)$, where $\text{dist}_G(u, x) = 1$ and $\text{dist}_H(v, y) = 2$. Now consider the level diagram of $G \times H$ with $(u, v)$ at level zero. It is easy to see that $(u, y)$ is at level two. Further, since $a_1(G) > 0$, the vertex $(x, y)$ is also at level two. However, the number of common neighbors of $(u, v)$ and $(u, y)$ is $\Delta(G) \cdot \Delta(H)$, while the number of common neighbors of $(u, v)$ and $(x, y)$ is $a_1(G) \cdot \Delta(H)$. Since $a_1(G) < \Delta(G)$, it follows that $G \times H$ is not distance-regular. The converse follows from Theorem 2.2.

The reader may check to see that results 2.1, 2.2, 2.3, and 2.4 lead to the following. If $G$ and $H$ are distance-regular graphs of diameter at least two, then $G \times H$ or a component of $G \times H$ is distance-regular iff each of $G$ and $H$ is isomorphic to $K_{n,n}$ for some $n$.

By an argument as in the proof of Theorem 2.4, if $G$ is a distance-regular graph with $d(G) \geq 2$ and $n \geq 3$, then $G \times K_n$ is not distance-regular. The following result takes care of $K_m \times K_n$.

**Theorem 2.5.** For $m, n \geq 3$, $K_m \times K_n$ is distance-regular iff $m = n$.

**Proof.** Let $m, n \geq 3$, and consider the graph $K_m \times K_n$ which is a nonbipartite, regular graph of diameter two and degree $(m - 1) \cdot (n - 1)$. Let us examine the level diagram of $K_m \times K_n$ with vertex $(0, 0)$ at level zero.

Vertices at level one are of the form $(i, j)$, where $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$. Thus, this level has a total of $(m - 1) \cdot (n - 1)$ vertices. The remaining $m + n - 2$ vertices are at level two, and are of the form $(i, 0)$ and $(0, j)$, where $1 \leq i \leq m - 1$ and $1 \leq j \leq n - 1$.

A vertex $(i, j)$ at level one is adjacent to a vertex $(p, q)$ at the same level iff

(i) $1 \leq p \leq m - 1$, $p \neq i$, and

(ii) $1 \leq q \leq n - 1$, $q \neq j$.

It follows that $a_1 = (m - 2) \cdot (n - 2)$. Since $c_1 = 1$, we have $b_1 = m + n - 4$. (Recall that $a_i + b_i + c_i = \Delta$.)

Next examine adjacency among elements at level two. If a vertex is of the form $(i, 0)$ (respectively, $(0, j)$), then it has a total of $n - 1$ (respectively, $m - 1$) neighbors on that level. Based on this, we have $a_2$ and $c_2$ in Table 1.

<table>
<thead>
<tr>
<th>For a vertex of the form $(i, 0), 1 \leq i \leq m - 1$</th>
<th>For a vertex of the form $(0, j), 1 \leq j \leq n - 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_2$</td>
<td>$n - 1$</td>
</tr>
<tr>
<td>$c_2$</td>
<td>$(m - 2) \cdot (n - 1)$</td>
</tr>
</tbody>
</table>

It follows that the numbers $a_k$, $b_k$, and $c_k$ depend only on $k$ and not on the choice of a vertex $(i, j)$ iff $m = n$. Note also that the level diagram itself is independent of the choice of the vertex at level 0.

**Dealing with $G \times K_2$**

In the rest of the paper, we present certain remarks with respect to $G \times K_2$. This graph (that is connected iff $G$ is nonbipartite) has been called bipartite double of $G$ by Brouwer et al. who present a characterization for its distance-regularity and other related results [2, pp. 24-26]. In particular, they prove the following.

(1) $G \times K_2$ is distance-regular of diameter $2d + 1$ iff $G$ is distance-regular with $a_i = 0$ ($i < d$) and $a_d > 0$. In this case, $G \times K_2$ is an antipodal 2-cover of $G$.

(2) If $G$ is distance-regular and $j = \min \{i \mid a_i \neq 0\} < d$, then $G \times K_2$ is distance-regular iff $d = 2j$, $a_j = c_{j+1}$, $b_{j-i} = c_{j+i+1} = c_{j+i} + a_{j+i}$ $(i = 1, \ldots, j - 1)$; if this is the case, then $d(G \times K_2) = 2j + 1 = d + 1$. 
Odd cycles, complete graphs (on at least three vertices), Petersen graph, and Hoffman-Singleton graph are certain examples that satisfy conditions in (1), while Shrikhande graph with intersection array $[(6, 3); (1, 2)]$ and Clebsch graph (that is isomorphic to the halved 5-cube) are examples that satisfy conditions in (2).

An Example

There exists a graph $G$ such that $G$ is not distance-regular, yet $G \times K_2$ is distance-regular. To see this, consider the graph that appears in Figure 1. The reader may check to see that

(i) $G$ is not distance-regular, and
(ii) $G \times K_2$ is isomorphic to $Q_4$ (appearing in Figure 2) that is known to be distance-regular.

![Figure 1. Graph G.](image1)

![Figure 2. The graph $Q_4$.](image2)

REFERENCES