Remarks on the Upper Bound on the Bisection Width of a Diagonal Mesh

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Abstract: A simple and intuitive proof is presented for Tang and Kamoua’s upper bound on the bisection width of a diagonal mesh. Further, an upper bound is presented for a related graph that is the Kronecker product of an odd cycle and an even cycle.

Key words: Bisection width; diagonal mesh; Kronecker product; cycle.

The bisection width of a connected graph is defined to be the least number of edges whose removal results in two components having an equal number (plus/minus one) of vertices. Tang and Padubidri [5] earlier studied a network called a diagonal mesh and presented an upper bound on its bisection width. I reported an error in their argument [2]. In response, Tang and Kamoua [4] recently revised the upper bound. While their result is correct, the proof (by induction on pp. 430-431) is rather complicated. The objective of this note is to present a proof that is substantially simpler and intuitive.

It is not difficult to see that an $m \times n$ diagonal mesh (where $m$ and $n$ are both odd and greater than or equal to 3) is isomorphic to the Kronecker product (defined below) of $C_m$ and $C_n$.

For $m \geq 2$, let $P_m$ denote a path on $m$ vertices, where $V(P_m) = \{0, \ldots, m-1\}$, and where adjacencies are defined in a natural way. Similarly, let $C_n$ denote a cycle on $n$ vertices where $V(C_n) = \{0, \ldots, n-1\}$. Further, let $bw(G)$ denote the bisection width of a connected graph $G$. The Kronecker product $G \times H$ of graphs $G = (V, E)$ and $H = (W, F)$ is defined as follows: $V(G \times H) = V \times W$ and $E(G \times H) = \{(a,x), (b,y)\}: \{a,b\} \in E$ and $\{x,y\} \in F$. This product is variously known as direct product, cardinal product and tensor product [1]. It is commutative and associative in a natural way.
is easy to see that $|V(G \times H)| = |V| \cdot |W|$ and $|E(G \times H)| = 2 \cdot |E| \cdot |F|$. Here are some relevant properties of $P_m \times P_n$ and $C_m \times C_n$ [1]:

1. Each of $P_m \times P_n$ and $C_{2i} \times C_{2j}$ consists of two connected components while $C_{2i+1} \times C_n$ is connected.

2. Graphs $P_m \times P_n$ and $C_m \times C_{2j}$ are bipartite while $C_{2i+1} \times C_{2j+1}$ is nonbipartite.

3. Graphs $P_m \times P_n$ and $C_{2i} \times C_{2j}$ are such that vertices $(p, q)$ and $(r, s)$ belong to the same connected component if and only if $p + q$ and $r + s$ are of the same parity. Thus it is convenient to refer to the two components of $P_m \times P_n$ (resp. $C_{2i} \times C_{2j}$) as even component and odd component, respectively. (Vertex set of $P_9 \times P_5$ appears in Figure 1.)

\begin{figure}[h]
\centering
\begin{tabular}{ccccccc}
00 & 02 & 04 & 01 & 03 \\
11 & 13 & 10 & 12 & 14 \\
20 & 22 & 24 & 21 & 23 \\
31 & 33 & 30 & 32 & 34 \\
40 & 42 & 44 & 41 & 43 \\
51 & 53 & 50 & 52 & 54 \\
60 & 62 & 64 & 61 & 63 \\
71 & 73 & 70 & 72 & 74 \\
80 & 82 & 84 & 81 & 83 \\
\end{tabular}
\caption{(a) Even Component \hspace{1cm} (b) Odd Component}
\end{figure}

It is easy to see that the even component of $P_m \times P_n$ consists of $\lceil mn/2 \rceil$ vertices while the odd component consists of $\lfloor mn/2 \rfloor$ vertices.

**Theorem 1** If $m$ and $n$ are odd integers and $m > n \geq 3$, then $bw(C_m \times C_n) \leq 2m$. 

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Proof. Let $m$ and $n$ be as stated. For the graph $P_m \times P_n$ (or $C_m \times C_n$), let $V_e$ and $V_o$ denote the vertex set of its even component and the vertex set of its odd component, respectively. It is clear that $V_e \cup V_o$ constitutes a vertex partition of $C_m \times C_n$ into two subsets of equal size (plus/minus one). Let $k = \frac{1}{2}(n + 1)$. Idea is to select a set of $k^2$ vertices in $V_e$ and a set of as many vertices in $V_o$, and swap them between $V_e$ and $V_o$; the resulting partition leads to the desired result.

Consider the following subset of $V_e$:

$$S_e = \{(i, j) : i + j \text{ is even and } i + j \leq n - 1\}.$$ 

Vertices in this set are precisely those appearing in the left $k$ “diagonals” of $V_e$. (See Figure 2(a) where $k = 3$.) Observe that $|S_e| = 1 + 3 + \cdots + 2k - 1 = k^2$. Next consider the following subset of $V_o$:

$$S_o = \{(i, j) : i + j \text{ is odd and } i + j \leq n - 2\} \cup \{(n, 0), (n - 1, 1), \ldots, (n - k + 1, k - 1)\}.$$ 

Vertices in this set are essentially those appearing in the left $k - 1$ “diagonals” of $V_o$ plus $k$ elements in the $k$th diagonal. (See Figure 2(b).) Observe that $|S_o| = (2 + 4 + \cdots + (2k - 2)) + k = k^2$.

Let $W_e = (V_e \setminus S_e) \cup S_o$ and $W_o = (V_o \setminus S_o) \cup S_e$. It is clear that $|W_e| = \lceil mn / 2 \rceil$.
and \(|W_o| = \lfloor mn/2 \rfloor\). I claim that there are exactly \(2m\) edges in \(C_m \times C_n\) having one end-point in \(W_e\) and the other in \(W_o\). To that end, such edges may be classified into

(a) a path of length \(m - n - 1\), (b) a path of length \(m - n + 1\) and (c) a set of \(2n\) edges.

The two paths are as follows:

- \((0, 0) - (m - 1, n - 1) - (m - 2, 0) - (m - 3, n - 1) - \cdots - (n + 2, 0) - (n + 1, n - 1)\),

and

- \((1, 0) - (0, n - 1) - (m - 1, 0) - (m - 2, n - 1) - \cdots - (n + 1, 0) - (n, n - 1)\).

The remaining \(2n\) edges are as follows:

- \((n, 0) - (n + 1, 1)\)
- \((n - 1, 0) - (n, 1)\) and \((n - 1, 1) - (n, 2)\)
- \((n - 2, 1) - (n - 1, 2)\) and \((n - 2, 2) - (n - 1, 3)\)
  
  \vdots
- \((n + 1, n - 3) - (n + 3, n - 2)\) and \((n + 1, n - 1) - (n + 3, n + 1)\)
- \((n + 1, n - 2) - (n + 1, n + 1)\)
- \((n - 1, n - 2) - (n + 1, n + 1)\) and \((n - 3, n - 2) - (n + 1, n + 3)\)
  
  \vdots
- \((1, n - 2) - (2, n - 1)\) and \((0, n - 2) - (1, n - 1)\).

The vertex partition of \(C_9 \times C_5\) based on the present argument has been illustrated

in Figure 3 where elements of \(W_e\) appear in “ovals.” Only those edges have been shown

that run between \(W_e\) and \(W_o\).

**Theorem 2** If \(m\) is odd, then \(bw(C_m \times C_m) \leq 2m + 2\).

**Proof.** The argument is similar to that in the proof of Theorem 1: (i) Start from

the natural partition \(V_e \cup V_o\), (ii) identify a set of \(k^2\) vertices in each set (where \(k = \lfloor \frac{1}{2}(m + 1) \rfloor\)) and swap them, leading to a partition \(W_e \cup W_o\) of \(C_m \times C_m\). It turns out

that there are exactly \(2m + 2\) edges between \(W_e\) and \(W_o\).

In the remainder of this note, I present an upper bound on \(bw(C_m \times C_n)\) where \(m\)

and \(n\) are not both odd.
Theorem 3 If \( m \) and \( n \) are both even, then the bisection width of each component of \( C_m \times C_n \) is less than or equal to \( \min\{2m, 2n\} \).

Proof. Let \( m \) and \( n \) be both even, \( m \geq n \). The two connected components of \( C_m \times C_n \) are mutually isomorphic [3], so consider the even component. There are two \( n \)-cycles in this graph whose removal results in two connected components, each having \( mn/4 \) vertices. The cycles themselves are as follows.

- \((0, 0) - (m - 1, 1) - (0, 2) - (m - 1, 3) - \cdots - (m - 1, n - 1) - (0, 0)\), and
- \((m/2, 0) - (m/2 - 1, 1) - (m/2, 2) - (m/2 - 1, 3) - (m/2 - 1, n - 1)\).

The construction is illustrated in Figure 4 with respect to the even component of \( C_8 \times C_6 \).

Corollary 4 If \( m \) is odd and \( n \) is even, then the bisection width of \( C_m \times C_n \) is less than or equal to \( \min\{4m, 2n\} \).
Figure 4: Vertex partition of the even component of $C_8 \times C_6$

Proof. Use the fact that $C_{2i+1} \times C_{2j}$ is isomorphic to each component of $C_{4i+2} \times C_{2j}$ [3] and appeal to Theorem 3.

Here is a summary of the results of this note.

\[ bw(C_m \times C_n) \leq \begin{cases} 
\min\{4m, 2n\}, & \text{if } m \text{ is odd and } n \text{ is even} \\
\max\{2m, 2n\}, & \text{if } m \text{ and } n \text{ are both odd and } m \neq n \\
2m + 2, & \text{if } m \text{ and } n \text{ are both odd and equal.}
\end{cases} \]

References


