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GLAS-PPE/01–04
3rd April 2001

A Tevatron Kinematics Primer

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Abstract

Presented here is an outline of the kinematics of hard-scattering processes in hadron-hadron collisions, with emphasis placed on the $p\bar{p}$ collisions produced by the Fermilab Tevatron collider.

Relata refero

1 Introduction

The majority of the $p\bar{p}$ collisions produced by the Fermilab Tevatron collider involve only small momentum transfers between the constituent partons of each hadron, and therefore fall outside the confines of perturbative QCD. However, there are collisions where the momentum transfer is sufficiently large enough such that the interaction between the partons can be predicted using a low-order QCD calculation.

It is this phenomenon of hard-scattering, which is, by definition, a process where the momentum transfer is greater than 1 GeV, that is of paramount interest. We consider the generic $2 \rightarrow 2$ process of the following hadronic interaction,

$$1 + 2 \rightarrow 3 + 4 \tag{1}$$

in terms of the parton model (Section 2). This allows the construction of the relevant parton 4-momenta, from which we can calculate the Lorentz boost (β^*) and Lorentz gamma factor (γ^*), and thus transform between the rest (Lab) frame and the centre of mass (CoM) frame (Section 3).

The relation between these two reference frames, the Lorentz Transformation, is then used to introduce a new quantity called rapidity (and the associated quantity pseudorapidity), which is an important concept in Experimental Particle Physics (Section 4). Finally, the Mandelstam variables are introduced so that the cross-section can be expressed in terms of physically observable quantities (Section 5).

2 The Parton Model

The parton model states that we can consider high-energy hadrons as being comprised of quasi-free particles (partons) that collectively hold the momentum of the hadron [1]. Therefore, a hadron of momentum $\mathbf{p}$ can be thought of as a collection of partons of longitudinal momentum $x_i\ \mathbf{p}$[1], where the momentum fraction x_i satisfies the following relations:

$$\begin{aligned} &0 \leq x_i \leq 1 \\ &\Sigma_i\ x_i = 1\ , \end{aligned} \tag{2}$$

In terms of the co-ordinate system used at CDF, the longitudinal component of the momentum lies along the z-axis. The generic lowest-order hadron-hadron interaction is depicted in Figure 1 [2],

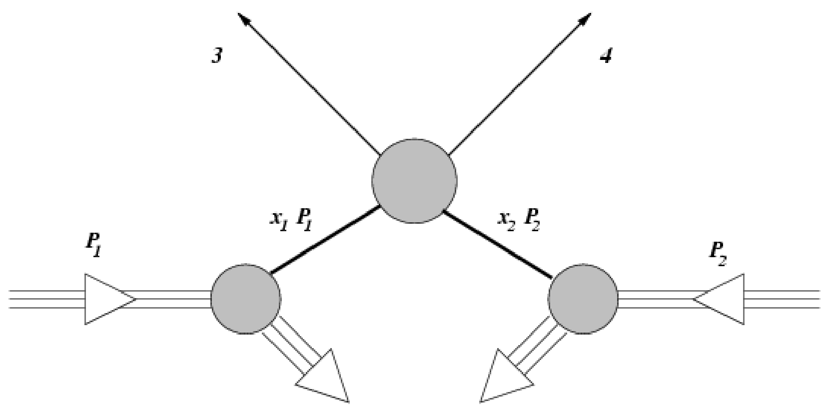

Figure 1: A generic two-body parton scattering process.

the cross-section of which is:

$$d\sigma(p\bar{p} \to 3+4) = \Sigma_{12}\ x_1 f_p(x_1)\ x_2 f_{\bar{p}}(x_2)\, d\sigma(1+2 \to 3+4) \tag{3}$$

where $f_h(x_i)$ is the probability of finding a parton of momentum fraction x_i in our initial state hadron, h. The cross-section is considered in more detail in Section 5.

[1] The assumption that the initial transverse momenta of the partons, typically $\mathcal{O}(300\text{ MeV})$, is negligible in comparison to the longitudinal momenta is adequate for our purposes.

3 4-Momenta

Now, consider the momenta 4-vectors of our initial-state partons, p_1 and p_2. Assume that the transverse momentum of each parton can be safely neglected, and that the mass of the proton is small in comparison to the energy, E_{beam}, of the parton. i.e.,

$$\begin{aligned} p_x &= p_y = 0, \\ p_z &= x_i \, \mathbf{p}, \\ \text{and} \quad \mathbf{p} &= E_{beam} \ . \end{aligned} \tag{4}$$

Thus, the 4-momenta of our initial-state partons are:

$$\begin{aligned} p_1 &= (x_1 \ E_{beam}, \ 0, \ 0, x_1 \ E_{beam}), \\ p_2 &= (x_2 \ E_{beam}, \ 0, \ 0, -x_2 \ E_{beam}). \end{aligned} \tag{5}$$

From the 4-momenta of the initial-state partons, we obtain the 4-momentum of the CoM frame[2)], p^*,

$$p^* = ((x_1 + x_2) \ E_{beam}, \ 0, \ 0, (x_1 - x_2) \ E_{beam}) \ , \tag{6}$$

and hence the Lorentz boost, in terms of the initial-state partons,

$$\beta^* = \frac{p_z}{E} = \frac{x_1 - x_2}{x_1 + x_2} \ . \tag{7}$$

The Lorentz gamma factor, γ^*, can also be expressed in similar terms by substituting Equation (7) into:

$$\gamma = \frac{1}{\sqrt{(1 - \beta^2)}} \ . \tag{8}$$

Thus,

$$\gamma^* = \frac{x_1 + x_2}{2\sqrt{x_1 x_2}} \tag{9}$$

The Lorentz Transform between the laboratory and the CoM frame can now be formulated in terms of the parton model in a familiar form:

$$\begin{aligned} E^* &= \gamma^*(E - \beta^* p_z) \\ p_z^* &= \gamma^*(p_z - \beta^* E) \ . \end{aligned} \tag{10}$$

We are now in a position to introduce a new quantity: *rapidity*.

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2) We define the CoM frame as being the rest frame of the hard-scattering subsystem.

4 Rapidity

Rapidity is often the quantity of choice of the Experimental Particle Physicist for measuring polar angles because rapidities are additive under successive Lorentz boosts in the same direction, and rapidity differences are Lorentz invariant (frame independent).

4.1 Definitions:

i. Rapidity

Since the hyperbolic tangent function always lies between -1 and $+1$, it is sensible to define velocities along the z-axis (β) in such terms, i.e.

$$\tanh y = \beta \ , \tag{11}$$

This is the definition of rapidity, y. However, the rapidity is more commonly expressed in the following way:

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \ , \tag{12}$$

which can be obtained using the inverse hyperbolic definition of tanh,

$$\tanh^{-1} z = \frac{1}{2} \ln \frac{1 + z}{1 - z} \ . \tag{13}$$

The Lorentz gamma factor can also be obtained in terms of rapidity by substituting Equation (11) into Equation (8), which gives us:

$$\cosh y = \gamma \ . \tag{14}$$

By combining Equations (11) and (14), we obtain the following:

$$\sinh y = \beta\gamma \ . \tag{15}$$

We can now rewrite Equations (10) in terms of rapidity by using the identities from Equations (14) and (15):

$$\begin{aligned} E^* &= \cosh y \ E - \sinh y \ p_z \\ p_z^* &= -\sinh y \ E + \cosh y \ p_z \ . \end{aligned} \tag{16}$$

$$\begin{aligned} c.f. \ \ x' &= \cos\theta \ x + \sin\theta \ y \\ y' &= -\sin\theta \ x + \cos\theta \ y \ . \end{aligned} \tag{17}$$

As we can see, this is mathematically analogous to a rotation about an axis. Just as successive rotations about the same axis, θ_1 followed by θ_2, say, are additive ($\theta_{total} = \theta_1 + \theta_2$), successive rapidity boosts *in the same direction*, y_1 followed by y_2, say, are also additive ($y_{total} = y_1 + y_2$).

ii. Pseudorapidity

The quantity *pseudorapidity* is a handy approximation to the rapidity when the mass of the particle can be assumed to be small in comparison to the energy. It is typically used when either the mass or the momentum of the particle is unknown. In this case, it is necessary to express β in terms of the scattering angle , θ (see Figure 2), i.e.

$$\beta = \cos\theta \ . \tag{18}$$

Substituting this expression into Equation (12):

$$y = \frac{1}{2}\ln\frac{1+\cos\theta}{1-\cos\theta} \ . \tag{19}$$

Using the following t-formula identity, where $t = \tan\frac{\theta}{2}$

$$\cos\theta = \frac{1-t^2}{1+t^2} = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} \ , \tag{20}$$

and some trigonometric manipulation, we obtain an expression for the pseudorapidity, which we denote as η in order to distinguish it from the rapidity:

$$\eta = -\ln\tan\frac{\theta}{2} \ . \tag{21}$$

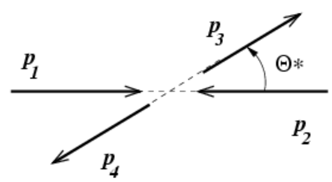

Figure 2: The Scattering Angle, θ^*, in the CoM frame.

iii. Notes on Rapidity and Pseudorapidity

Through a combination of laziness and the sloppy use of the English language, it is not uncommon for the terms rapidity and pseudorapidity to be used interchangeably. **This is wrong!** Pseudorapidity is an approximation to rapidity when either the mass of a particle can be assumed to be small in comparison to its energy, or the mass and/or momentum of the particle is unknown. It is therefore common that pseudorapidity is correctly used, and denoted appropriately as η, but is still referred to as rapidity[3)]. For the duration of this chapter, we consider rapidities; only in subsequent chapters will pseudorapidity be considered, and only when we have justification for doing so.

[3)] this is probably because pseudorapidity is shorter than rapidity, which also explains why pseudorapidity is often referred to as *eta*!

4.2 Lorentz Transformation

We now want to be able to move freely between the Lab frame and the CoM frame in terms of rapidity. To do this, we start with Equation (12) in the CoM frame, i.e.

$$y^* = \frac{1}{2} \ln \frac{E^* + p_z^*}{E^* - p_z^*} \tag{22}$$

and substitute in our expressions for E^* and p_z^* from Equations (10). Expanding out and rearranging:

$$\Rightarrow y^* = \frac{1}{2} \ln \frac{\gamma^*(E - \beta^* p_z) + \gamma^*(p_z - \beta^* E)}{\gamma^*(E - \beta^* p_z) - \gamma^*(p_z - \beta^* E)}$$

$$\Rightarrow y^* = \frac{1}{2} \ln \frac{\gamma^*(E - \beta^* p_z + p_z - \beta^* E)}{\gamma^*(E - \beta^* p_z - p_z + \beta^* E)}$$

$$\Rightarrow y^* = \frac{1}{2} \ln \frac{(E + p_z)(1 - \beta^*)}{(E - p_z)(1 + \beta^*)} \ . \tag{23}$$

By using the properties of logarithms, we obtain the intermediate result:

$$y^* = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \ - \ \frac{1}{2} \ln \frac{1 + \beta^*}{1 - \beta^*} \ . \tag{24}$$

This can be condensed further:

$$y^* = y - y_{boost} \ . \tag{25}$$

where y is the rapidity in the Lab frame, y^* is the rapidity in the CoM frame, and y_{boost} is the boost to the rapidity required to move from one frame to the other.

Equations (24) and (25) have been written in this form so that y_{boost} will be positive when $x_1 \geq x_2$.

4.3 y_3 and y_4

In the CoM frame, the rapidities of partons 3 and 4 will be equal and opposite, thus

$$y_3{}^* = -y_4{}^* = y^* \ . \tag{26}$$

The decision to take $y_3{}^*$ as being positive is purely arbitrary and is just a question of convention.

Using the above formula with Equation (25), we are now in a position to write down explicitly the relationship between the Lab frame and the CoM frame in terms of the rapidities of each of the final-state partons. Consider first final-state parton 3,

$$\begin{aligned} y_3{}^* &= y_3 - y_{boost} \\ \Rightarrow y_3 &= y_3{}^* + y_{boost} \end{aligned} \tag{27}$$

Similarly,

$$\begin{aligned} -{y_4}^* &= y_4 - y_{boost} \\ \Rightarrow y_4 &= y_{boost} - {y_4}^* \end{aligned} \qquad (28)$$

It is not uncommon for the subscripts 3 and 4 to be dropped from the ${y_i}^*$ terms, so Equations (27) and (28) are often written as:

$$\begin{aligned} y_3 &= y^* + y_{boost} \\ y_4 &= y_{boost} - y^* \end{aligned} \qquad (29)$$

By rearranging Equations (29), we can obtain the following expressions for y^* and y_{boost} in terms of y_3 and y_4:

$$\begin{aligned} y^* &= \frac{1}{2}(y_3 - y_4) \\ y_{boost} &= \frac{1}{2}(y_3 + y_4) \end{aligned} \qquad (30)$$

With expressions for the rapidities of partons 3 and 4 in both the Lab frame and the CoM frame, we can now see that the difference between the rapidities is $2y_*$, regardless of the frame, i.e. rapidity differences are invariant.

4.4 Example of a Rapidity Boost

Figure 3 illustrates the rapidities of the final-state parton 3 and 4 in both the Lab frame and the CoM frame, and the rapidity boost between the two.

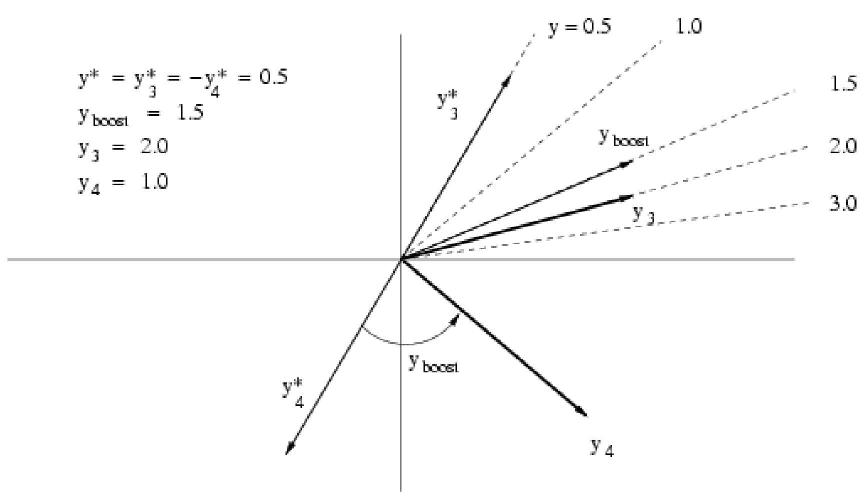

Figure 3: A Rapidity Boost between the Lab frame and the CoM frame.

5 Cross-Section

We assume that the mass of the partons are small in comparison to the energy, therefore pseudorapidity is used instead of rapidity. In parton terms, the cross-section for the generic $2 \to 2$ process is:

$$\frac{d^3\sigma}{dx_1 dx_2 d\hat{t}}(p\bar{p} \to 3+4) = f_1(x_1) f_2(x_2) \frac{d\hat{\sigma}}{d\hat{t}}(1+2 \to 3+4) \; , \tag{31}$$

where $\hat{t}$ is a measure of the 4-momentum transfered (See § 5.1). It is necessary to translate this expression for the cross-section into a form that contains physically observable parameters of the final state. The three physically observable parameters in the final state are chosen to be the transverse momenta, $p_\perp$, of the partons and the *pseudo*rapidities, η_3, η_4, of the final-state partons. It will shown that the set of variables (η_3, η_4, $p_\perp$) are related in a straight-forward manner to the underlying parton variables, (x_1, x_2, $\hat{t}$).

5.1 The Mandelstam Variables

We first define three new quantities, known collectively as the Mandelstam variables:

$$\begin{aligned} \hat{s} &= (p_1 + p_2)^2 = (p_3 + p_4)^2 \\ \hat{t} &= (p_1 - p_3)^2 = (p_2 - p_4)^2 \\ \hat{u} &= (p_1 - p_4)^2 = (p_2 - p_3)^2 \end{aligned} \tag{32}$$

The Mandelstam variables are Lorentz invariant quantities that are used extensively to describe the kinematics of particle reactions. The physical interpretation of these quantities are as follows: $\hat{s}$ is the CoM energy squared of the hard-scattering subsystem[4)]; $\hat{t}$ is the squared 4-momentum transfered between the initial-state parton 1 and the final-state parton 3; and $\hat{u}$ is the 4-momentum squared transfered between the initial-state parton 1 and the final-state parton 4.

Note 1: $\hat{t}$ and $\hat{u}$ are similar. In fact, they essentially represent the squared 4-momentum transfered in the forward and backward directions respectively.
Note 2: It is the *4-momentum* that is used, not the 3-momentum. In a purely elastic case, however, the energy would remain unchanged, so you would therefore only be required to consider the 3-momentum.

The Mandelstam variables may be written in the following way:

$$\begin{aligned} \hat{s} &= x_1 x_2 s \\ \hat{t} &= -\frac{1}{2}\hat{s}(1 - \cos\theta^*) \\ \hat{u} &= -\frac{1}{2}\hat{s}(1 + \cos\theta^*) \end{aligned} \tag{33}$$

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4) c.f. the CoM energy squared of the $p\bar{p}$ system, $s = 4E_{beam}^2$

5.2 Miscellaneous Equations

Before we consider the cross-section, we need equations for x_1, x_2, E and p_z in terms of pseudorapidity, for reasons that will become clear in the following section.

i. x_1 and x_2

The pseudorapidity boost, η_{boost}, can be written in terms of x_1 and x_2 by inserting Equation (7) into:

$$\eta_{boost} = \frac{1}{2} \ln \frac{1+\beta^*}{1-\beta^*} \ . \tag{34}$$

Therefore:

$$\eta_{boost} = \frac{1}{2} \ln \frac{x_1}{x_2} \ . \tag{35}$$

Using this expression for the rapidity boost and the Mandelstam variable $\hat{s}$, which we recall is the CoM energy squared of our hard-scattering subsystem, we can determine x_1 and x_2:

$$\begin{aligned} x_1 &= \sqrt{\frac{\hat{s}}{s}} \ \exp(\eta_{boost}) \ , \\ x_2 &= \sqrt{\frac{\hat{s}}{s}} \ \exp(-\eta_{boost}) \ . \end{aligned} \tag{36}$$

ii. E and p_z

Re-arranging Equation (21), we obtain the following expression for t:

$$t \ = \ -\exp(\eta) \ . \tag{37}$$

By substituting the appropriate t-formulae into the following equations,

$$\begin{aligned} E_\perp = \ E \ \sin\theta \ &= \ E \ \frac{2t}{1+t^2} \ , \\ p_\perp = \ p_z \ \tan\theta \ &= \ p_z \ \frac{2t}{1-t^2} \ , \end{aligned} \tag{38}$$

and then evaluating using Equation (37), we obtain expressions for the energy and longitudinal momentum of any given final-state parton in terms of the transverse momentum and the pseudorapidity:

$$\begin{aligned} E &= p_\perp \cosh\eta \ , \\ p_z &= p_\perp \sinh\eta \ . \end{aligned} \tag{39}$$

5.3 The Cross-Section in Physically Observable Terms

In the CoM frame of the hard-scattering subsystem, the total energy is $\sqrt{\hat{s}}$ and, by definition, the momentum of the final-state partons share this energy equally. Therefore,

$$\begin{aligned} p^*_{z\ 3} &= \frac{1}{2}\sqrt{\hat{s}}\cos\theta^* \\ p^*_{\perp\ 3} &= \frac{1}{2}\sqrt{\hat{s}}\sin\theta^* \ , \end{aligned} \tag{40}$$

with p_4 orientated exactly opposite.

To obtain a relationship between θ^* and η^*, substitute the above expression for the transverse momentum into the equation for energy in Equations (39).

$$\begin{aligned} E^*_3 &= (\frac{1}{2}\sqrt{\hat{s}}\sin\theta^*)\ \cosh\eta^* \\ \Rightarrow\ \frac{1}{\sin\theta^*} &= \cosh\eta^* \end{aligned} \tag{41}$$

The Mandelstam variables,

$$\begin{aligned} \hat{s} &= \frac{4p_\perp^2}{\sin^2\theta^*} \\ \hat{t} &= -\frac{1}{2}\hat{s}(1-\cos\theta^*) \ , \end{aligned} \tag{42}$$

can then be rewritten as:

$$\begin{aligned} \hat{s} &= 4p_\perp^2\cosh^2\eta^* \\ \hat{t} &= -2p_\perp^2\cosh\eta^*\exp(-\eta^*) \end{aligned} \tag{43}$$

Using this expression for $\hat{s}$, we can rewrite the equations for x_1 and x_2 given by Equations (36),

$$\begin{aligned} x_1 &= \frac{2p_\perp}{\sqrt{s}}\cosh\eta^*\exp(\eta_{boost}) \ , \\ x_2 &= \frac{2p_\perp}{\sqrt{s}}\cosh\eta^*\exp(-\eta_{boost}) \ . \end{aligned} \tag{44}$$

With expressions for x_1, x_2 and $\hat{t}$ in terms of η_3, η_4 and $p_\perp$, the cross-section can be translated into terms of physically observable quantities by using the Jacobian:

$$\frac{\partial(x_1,x_2,\hat{t})}{\partial(\eta_3,\eta_4,p_\perp)} = \frac{8p_\perp^3}{s}\cosh^2\eta^* = \frac{2p_\perp\hat{s}}{s} \ . \tag{45}$$

Multiplying Equation (31) by this factor gives:

$$\frac{d^3\sigma}{d\eta_3 d\eta_4 dp_\perp} = f_1(x_1)f_2(x_2)\frac{2p_\perp\hat{s}}{s}\frac{d\hat{\sigma}}{d\hat{t}}(1+2\to 3+4) \ . \tag{46}$$

This can be simplified further using the following relations:

$$\hat{s} = x_1 x_2 s \; ,$$
$$p_\perp dp_\perp = \frac{d^2 p_\perp}{2\pi} \; , \tag{47}$$

which gives our final result:

$$\frac{d^4\sigma}{d\eta_3 d\eta_4 d^2 p_\perp} = x_1 f_1(x_1) x_2 f_2(x_2) \frac{1}{\pi} \frac{d\hat{\sigma}}{d\hat{t}}(1+2 \to 3+4) \; . \tag{48}$$

5.4 Notes on the Cross-Section

In our final formula for the cross-section, Equation (48), x_1, x_2 and the Mandelstam variables of the hard-scattering subsystem are:

$$\begin{aligned} x_1 &= \frac{2p_\perp}{\sqrt{s}} \cosh\eta^* \exp(\eta_{boost}) \; , \\ x_2 &= \frac{2p_\perp}{\sqrt{s}} \cosh\eta^* \exp(-\eta_{boost}) \; . \\ \hat{s} &= 4p_\perp^2 \cosh^2\eta^* \\ \hat{t} &= -2p_\perp^2 \cosh\eta^* \exp(-\eta^*) \\ \hat{u} &= -2p_\perp^2 \cosh\eta^* \exp(+\eta^*) \end{aligned} \tag{49}$$

The cross-section is related to the matrix element, $\mathcal{M}$, which is the quantum-mechanical amplitude for a given process to occur. This is analogous to the scattering amplitude in non-relativistic quantum mechanics, so it follows that $|\mathcal{M}|^2$ is the probability with which a given process will occur. The relationship between the cross-section and the matrix element is:

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{\pi\alpha^2}{\hat{s}^2} \, |\mathcal{M}|^2 \; . \tag{50}$$

References

[1] E. Eichten, I. Hinchliffe, K. Lane and C. Quigg, Rev. Mod. Phys. **56** (1984) 579; Rev. Mod. Phys. **58** (1985) 1065

[2] M. Peskin and D. Schroeder, An Introduction to Quantum Field Theory.

Appendix A: Key Relations

Rapidities

$$y = \frac{1}{2} \ln \frac{E + p_z}{E - p_z} \tag{51}$$

$$\eta = -\ln \tan \frac{\theta}{2} \tag{52}$$

$$\begin{aligned} y_3 &= y^* + y_{boost} \\ y_4 &= y_{boost} - y^* \end{aligned} \tag{53}$$

$$\begin{aligned} y^* &= \frac{1}{2}(y_3 - y_4) \\ y_{boost} &= \frac{1}{2}(y_3 + y_4) \end{aligned} \tag{54}$$

Scattering Angle

$$\frac{1}{\sin \theta^*} = \cosh \eta^* \tag{55}$$

$$\cos \theta^* = \tanh \eta^* \tag{56}$$

$$\frac{1}{\tan \theta^*} = \sinh \eta^* \tag{57}$$

Mandelstam Variables

$$\begin{aligned} \hat{s} &= \frac{4p_\perp^2}{\sin^2 \theta^*} = x_1 x_2 s \\ \hat{t} &= -\frac{1}{2}\hat{s}(1 - \cos \theta^*) \\ \hat{u} &= -\frac{1}{2}\hat{s}(1 + \cos \theta^*) \end{aligned} \tag{58}$$

$$\begin{aligned} \hat{s} &= 4p_\perp^2 \cosh^2 \eta^* \\ \hat{t} &= -2p_\perp^2 \cosh \eta^* \exp(-\eta^*) \\ \hat{u} &= -2p_\perp^2 \cosh \eta^* \exp(+\eta^*) \end{aligned} \tag{59}$$

Appendix B: The Squared Matrix Element, $|\mathcal{M}|^2$

The squared matrix element, $|\mathcal{M}|^2$, describes the cross-section of various hard-scattering processes, and is defined in terms of the Mandelstam variables. The squared matrix element for some of the key hard-scattering processes are defined below:

$$q_1 q_2 \to q_1 q_2; \qquad |\mathcal{M}|^2 = \frac{4}{9}\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \tag{60}$$

$$q_1 q_1 \to q_1 q_1; \qquad |\mathcal{M}|^2 = \frac{4}{9}(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{s}^2 + \hat{t}^2}{\hat{u}^2}) - \frac{8}{27}\frac{\hat{s}^2}{\hat{u}\hat{t}} \tag{61}$$

$$q_1 \bar{q}_2 \to q_1 \bar{q}_2; \qquad |\mathcal{M}|^2 = \frac{4}{9}\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \tag{62}$$

$$q_1 \bar{q}_1 \to q_1 \bar{q}_1; \qquad |\mathcal{M}|^2 = \frac{4}{9}(\frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} + \frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2}) - \frac{8}{27}\frac{\hat{u}^2}{\hat{s}\hat{t}} \tag{63}$$

$$q_1 \bar{q}_1 \to gg; \qquad |\mathcal{M}|^2 = \frac{32}{27}\frac{\hat{t}^2 + \hat{u}^2}{\hat{t}\hat{u}} - \frac{8}{3}\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \tag{64}$$

$$gg \to q_1 \bar{q}_1; \qquad |\mathcal{M}|^2 = \frac{1}{6}\frac{\hat{t}^2 + \hat{u}^2}{\hat{t}\hat{u}} - \frac{3}{8}\frac{\hat{t}^2 + \hat{u}^2}{\hat{s}^2} \tag{65}$$

$$g q_1 \to g q_1; \qquad |\mathcal{M}|^2 = -\frac{4}{9}\frac{\hat{s}^2 + \hat{u}^2}{\hat{s}\hat{u}} + \frac{\hat{u}^2 + \hat{s}^2}{\hat{t}^2} \tag{66}$$

$$gg \to gg; \qquad |\mathcal{M}|^2 = -\frac{9}{2}(3 - \frac{\hat{t}\hat{u}}{\hat{s}^2} - \frac{\hat{s}\hat{u}}{\hat{t}^2} - \frac{\hat{s}\hat{t}}{\hat{u}^2}) \tag{67}$$

It can be seen that the squared matrix element for each process is independent of the CoM energy of the hard-scattering subsystem, and that it only depends on the scattering angle, θ. The squared matrix elements for each of the above processes as a function of the scattering angle are illustrated in Figures 4 and 5.

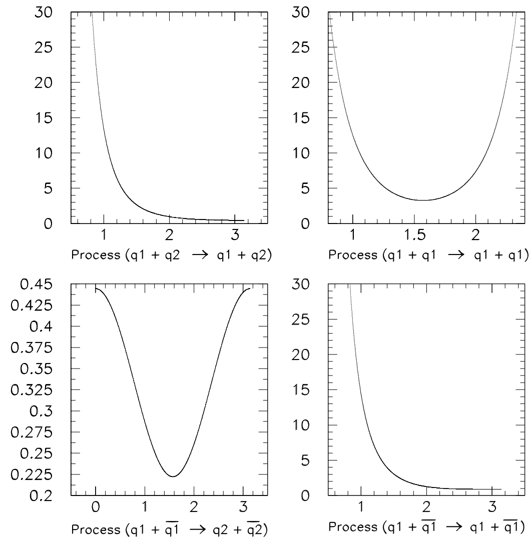

Figure 4: Plots of $|\mathcal{M}|^2$ as a function of θ.

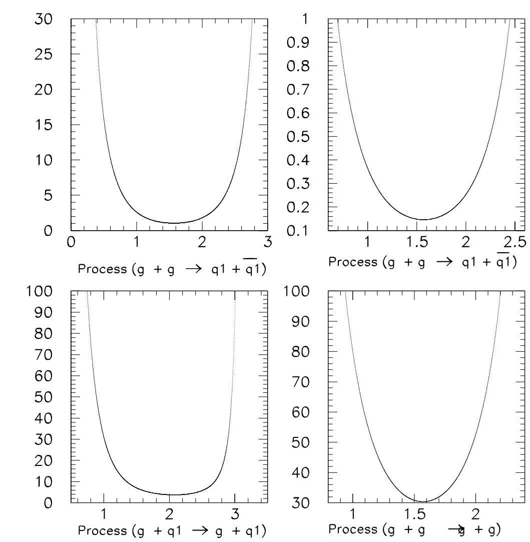

Figure 5: More Plots of $|\mathcal{M}|^2$ as a function of θ.