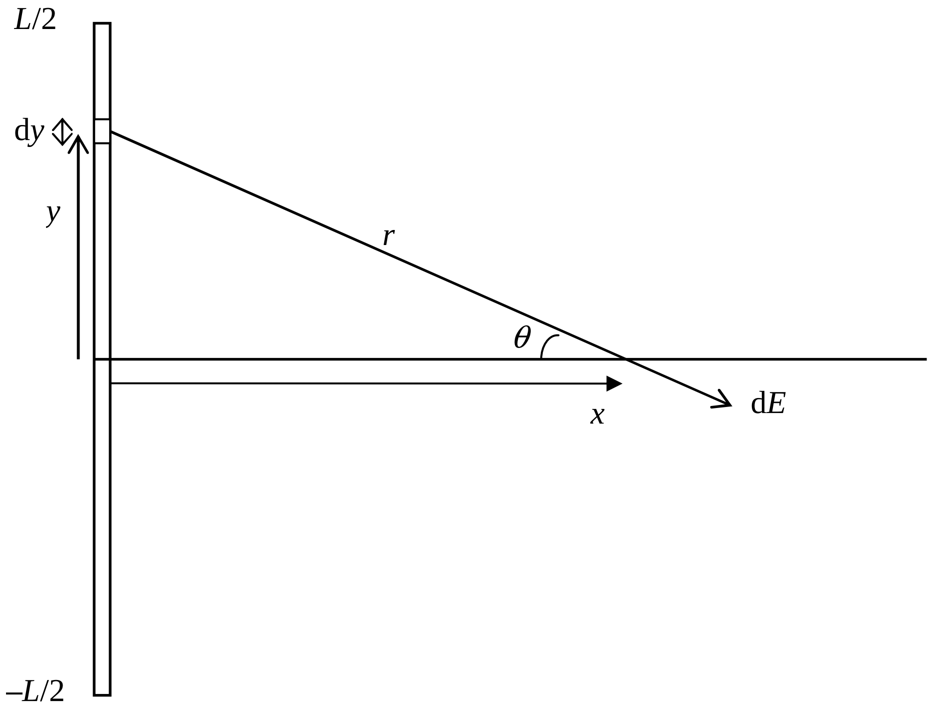
PHY101 Electricity and Magnetism I
Example Calculation (from lecture)

A rod of length L carries a charge Q distributed uniformly along its length. If it is centred on the origin and oriented along the y-axis, what is the resulting electric field at points on the x-axis?



The linear charge density along the rod is $\lambda = Q/L$.

To calculate the overall field at a point on the x-axis, we divide the rod into infinitesimal slices dy, find the contribution to E from each of these, and sum (integrate) using the principle of superposition.

Charge in slice dy *is* $\lambda\,\mathrm{d}y$.

$$\mathrm{d}E \text{ due to } \mathrm{d}y = \frac{\lambda\,\mathrm{d}y}{4\pi\varepsilon_0 r^2}.$$

By symmetry, the overall E field on the x-axis must be parallel to x. (Contributions in the negative y direction due to slices above the x-axis as shown will be cancelled by those in the positive y direction due to slices below the x-axis.)

$$\mathrm{d}E_x = \frac{\lambda\,\mathrm{d}y\cos\theta}{4\pi\varepsilon_0 r^2} \qquad \Rightarrow \qquad E = \int_{-L/2}^{L/2} \frac{\lambda\,\mathrm{d}y\cos\theta}{4\pi\varepsilon_0 r^2} \qquad (1)$$

We have now done all the physics, but have some work to do on the maths! The integral contains three variables, y, r and θ, which are all interdependent. There is also no explicit dependence on x, though we want to find the field a distance x from the origin. We must therefore change variables.

$$x = r\cos\theta \qquad \Rightarrow \qquad r = x/\cos\theta$$

$$y = x\tan\theta \qquad \Rightarrow \qquad \frac{\mathrm{d}y}{\mathrm{d}\theta} = x\sec^2\theta = \frac{x}{\cos^2\theta}\ (*) \qquad \text{or} \quad \mathrm{d}y = \frac{x}{\cos^2\theta}\mathrm{d}\theta$$

Substitute these into (1):

$$\Rightarrow \quad E = \frac{\lambda}{4\pi\varepsilon_0}\int \frac{\cos^2\theta}{x^2}\cos\theta \frac{x\mathrm{d}\theta}{\cos^2\theta} = \frac{\lambda}{4\pi\varepsilon_0 x}\int \cos\theta \mathrm{d}\theta = \frac{\lambda}{4\pi\varepsilon_0 x}[\sin\theta].$$

We now need to put in the correct limits. When $y = L/2$, $\sin\theta = \frac{L/2}{\sqrt{x^2+(L/2)^2}}$, and when $y = -L/2$, $\sin\theta = \frac{-L/2}{\sqrt{x^2+(L/2)^2}}$.

So $$E = \frac{\lambda}{4\pi\varepsilon_0 x}\left(\frac{L/2}{\sqrt{x^2+(L/2)^2}} - \frac{-L/2}{\sqrt{x^2+(L/2)^2}}\right) = \frac{Q}{4\pi\varepsilon_0 x\sqrt{x^2+(L/2)^2}}$$

The total electric field is $E = \frac{Q}{4\pi\varepsilon_0 x\sqrt{x^2+(L/2)^2}}$ in the x direction.

It is always useful to apply a "sanity check" (where this does not involve too much work!). The electric field at a large distance from the rod should not depend on the exact arrangement of the charge. If we consider the limit of the above expression when $x \gg L$, we obtain $E \to \frac{Q}{4\pi\varepsilon_0 x^2}$, exactly as expected!

(*) If you have forgotten that $\frac{\mathrm{d}\tan\theta}{\mathrm{d}\theta} = \sec^2\theta$, you can easily derive this using the identity $\tan\theta = \frac{\sin\theta}{\cos\theta}$ together with the quotient rule for differentiation.

For further calculations of electric field due to extended charge distributions, see the application section of the handout.