PHY101 Electricity and Magnetism I
Topic 2 (Lectures 2 & 3) – Electric Fields

In this topic, we will cover:
1) Electric Fields
2) Field Lines
3) Electric Fields and Conductors
4) Electric Dipoles in an Electric Field

Introduction

We have seen that Coulomb's law, like Newton's law of gravitation, involves *action at a distance* – one charge is affected by another with no physical contact between them. One way to envisage how this occurs is to consider that one charge modifies the properties of the space around it, and the other charge then responds to these modifications. This modification of the space is known as an electric field.

The Electric Field

Consider the electric field surrounding a static point charge Q. The field at a given position can be defined by the force produced by Q on a small test charge q_t at that point. The electric field strength $\mathbf{E}$ is equal to the force per unit charge at that point:

$$\mathbf{E} = \frac{\mathbf{F}}{q_t} \qquad [1]$$

From this definition, we can see that the S.I. unit for electric field strength is Newton per Coulomb (N C^{-1}). Note that $\mathbf{E}$ is a vector, with direction given by the force experienced by the (positive) test charge. Applying Coulomb's law for a point charge Q (see previous topic)

$$\mathbf{F} = \frac{Qq_t}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad [2]$$

We obtain the electric field about a point charge

$$\boxed{\mathbf{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}} \qquad [3]$$

Equation [3] is only valid for a point charge. For a system of point charges, we can apply the principle of linear superposition as we did before, calculating a resultant field strength which is equal to the vector sum of the field strengths due to each of the individual charges.

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2 + \mathbf{E}_3 + \cdots + \mathbf{E}_N = \sum \mathbf{E}_i \qquad [4]$$

Once the resultant field strength is known, the force on any charge q can be found from the definition of $\mathbf{E}$ given in [1], so

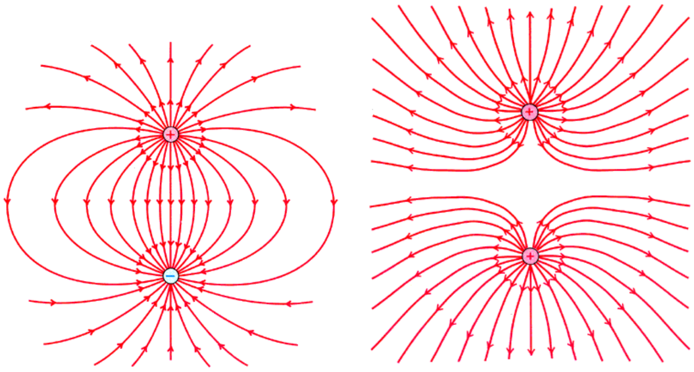
$$\mathbf{F} = q\mathbf{E}. \qquad [5]$$

Note that $\mathbf{E}$ is the field strength calculated from all *other* charges present, not including q itself! If q is positive, the direction of the force is that of the field strength; if q is negative, the direction of the force is opposite to the field.

Field Lines *or* Lines of Force

We can represent an electric field diagrammatically by drawing *field lines* or *lines of force*. These represent the direction of the electric field at each point in space, so show the direction a (small) free charge would tend to follow. Field lines obey the following rules:

1) Electrostatic field lines always start on positive charges and end on negative charges.
2) The number of lines originating from, or terminating on, a charge is proportional to the magnitude of the charge.
3) The direction of the field at a point is given by the tangent to the field line at that point.
4) The field strength is equal to the density of lines, that is the number of lines per unit area (where the area is on a surface normal to the field).



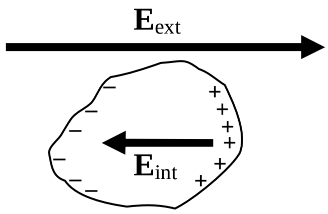
Opposite charges   Like charges

5) Lines of field never cross (since the direction of force on a test charge would be ill-defined at a point where lines crossed).

The diagram shows field lines around pairs of like and opposite charges.

Electric Field and Conductors

When a conductor is placed in a region of electric field, the free electrons within it respond to the force on their (negative) charge. If the applied external field is $\mathbf{E}_{ext}$, the electrons move in a direction opposite to $\mathbf{E}_{ext}$, as shown in the sketch. As they do so, they leave unbalanced positive charges behind, and these charges in the metal generate an internal field $\mathbf{E}_{int}$ opposite in direction to $\mathbf{E}_{ext}$. The electrons will continue to move until $\mathbf{E}_{int}$ = $\mathbf{E}_{ext}$; once this condition is met, the net electric field inside the conductor is zero, and no further charge flows.



Under static conditions, the net electric field within the material of a conductor is zero.

Note that the resulting field just outside the conductor will be the resultant of that due to the applied field and the displaced charges. Consider a resulting field at an angle to the surface of the conductor. We can resolve this into components parallel to and perpendicular to the local surface. The free electrons at the surface of the conductor will move in response to the component parallel to the surface, and quickly reduce this component to zero. Therefore the resulting field can only be perpendicular to the surface.

Under static conditions, the electric field at all points on the surface of a conductor is normal to that surface.

We shall shortly also show one other fact about conductors and electric charge:

Under static conditions, all the (unbalanced) electric charge resides on the surface of the conductor.

Continuous Charge Distributions

In the last lecture, we saw how the force, and therefore the field, due to a number of *point* charges can be evaluated using superposition. In order to find the field due to a *continuous* distribution, we must consider the charge divided up into infinitesimal elements dq which may be considered as point charges. The infinitesimal contribution to the total field produced by such an element is

$$d\mathbf{E} = \frac{dq}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}} \qquad [6]$$

where **r** is the vector from the charge element to the point where the field is considered. To determine the total field due to the distributed charge, the vector sum over the whole distribution of all such contributions must be evaluated by means of an integral:

$$\mathbf{E} = \int \frac{\mathrm{d}q}{4\pi\varepsilon_0 r^2}\hat{\mathbf{r}}\,. \qquad [7]$$

In practice, to evaluate the integral it is necessary to express dq in terms of r, as we will illustrate in the examples. We often need to consider charge distributed along a line, over a surface or throughout a volume. This leads to the concept of a variety of *charge densities*.

1) *Volume charge density* (or simply *charge density*) ρ. Units C m^{-3}. dq = ρ dV where dV is an element of volume.
2) *Surface charge density* σ. Units C m^{-2}. dq = σ dA where dA is an element of area.
3) *Linear charge density* λ. Units C m^{-1}. dq = λ dl where dl is an element of length.

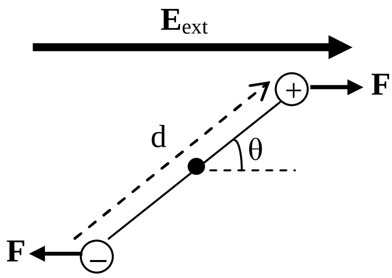
Electric Dipoles

A pair of equal and opposite charges separated by some distance is known as an *electric dipole*. If the charges have a magnitude Q and their separation is d, their dipole moment **p** is given by:

$$\boxed{\mathbf{p} = Q\mathbf{d}} \qquad [8]$$

where **d** is a vector pointing from the negative to the positive charge (so **p** points in the same direction).

When a dipole is placed in a uniform electric field **E**, it experiences no net force, as the two charges feel equal and opposite forces. However, since the two forces do not act through the same point, there is a moment or torque acting on the dipole. Consider the positive charge. It experiences a force $\mathbf{F} = Q\mathbf{E}$. The negative charge experiences an equal and opposite force, so the torque τ is given by one force multiplied by their perpendicular separation, $\tau = Fd\sin\theta = QEd\sin\theta = pE\sin\theta$.



The vector expression for the torque is thus

$$\boxed{\boldsymbol{\tau} = \mathbf{p}\times\mathbf{E}} \qquad [9]$$

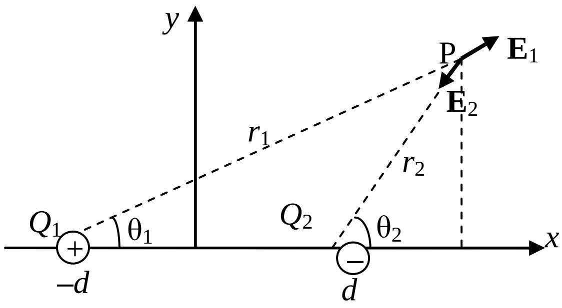
The dipole tends to align itself with an applied external electric field. This means that it takes work (provided by an opposing torque or pair of forces) to rotate the dipole against the field, resulting in an increasing potential energy of the dipole. Defining this potential energy as zero when **p** is perpendicular to **E**, we can see that in rotating to an angle θ as shown the dipole does work given by force×displacement = $F\,\frac{d}{2}\cos\theta$ against the applied force for *each* charge, so its potential energy is reduced by this amount. The overall potential energy of the dipole is thus

$$\boxed{U = -pE\cos\theta = -\mathbf{p}\cdot\mathbf{E}} \qquad [10]$$

Putting What You Have Learnt Into Practice

Question 1

One point charge Q_1 = 20 μC is placed at (–d, 0) and another Q_2 = –10 μC at (+d, 0) as shown. Find the resulting field strength at a point with coordinates (x,y), where d = 1.0 m and x = y = 2 m.



Solution

The charges, electric field vectors and coordinates are shown opposite.

By Pythagoras' theorem

$r_1 = \sqrt{\left(x+d\right)^2 + y^2} = \sqrt{13} = 3.6 \text{ m}$ $\qquad$ $r_2 = \sqrt{\left(x-d\right)^2 + y^2} = \sqrt{5} = 2.2 \text{ m}$

The magnitudes of the electric fields are therefore

$$E_1 = \frac{|Q_1|}{4\pi\varepsilon_0 r_1^2} = \frac{2\times10^{-5}}{4\pi\times8.85\times10^{-12}\times13} = 1.4\times10^4 \text{ N C}^{-1}$$

$$E_2 = \frac{|Q_2|}{4\pi\varepsilon_0 r_2^2} = \frac{1\times10^{-5}}{4\pi\times8.85\times10^{-12}\times5} = 1.8\times10^4 \text{ N C}^{-1}$$

The components of the resulting field strength, $\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_2$, are

$E_x = E_{1x} + E_{2x} = E_1 \cos\theta_1 - E_2 \cos\theta_2$ $\qquad$ $E_y = E_{1y} + E_{2y} = E_1 \sin\theta_1 - E_2 \sin\theta_2$.

From the diagram, $\sin\theta_1 = \dfrac{y}{r_1}$, $\sin\theta_2 = \dfrac{y}{r_2}$, $\cos\theta_1 = \dfrac{x+d}{r_1}$, $\cos\theta_2 = \dfrac{x-d}{r_2}$.

Hence

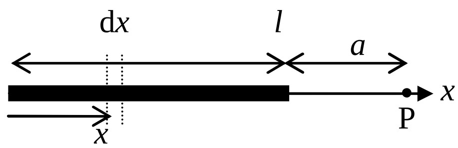
$$E_x = 1.4\times10^4 \frac{3}{3.6} - 1.8\times10^4 \frac{1}{2.2} = 3.5\times10^3 \text{ N C}^{-1}$$

$$E_y = 1.4\times10^4 \frac{2}{3.6} - 1.8\times10^4 \frac{2}{2.2} = -8.6\times10^3 \text{ N C}^{-1}$$

So $\qquad \boxed{\mathbf{E} = 3.5\times10^3\,\mathbf{i} - 8.6\times10^3\,\mathbf{j} \text{ N C}^{-1}}$

Question 2

A rod of length l has a total charge Q distributed uniformly along its length. Calculate the electric field at a point P located along the long axis of the rod and a distance a from one end.



Solution

Assume the rod is lying along the x-axis as shown, running from x = 0 to x = l. The charge density is given by $\lambda = \dfrac{Q}{l}$, and if an element has length dx it carries an elemental charge $\mathrm{d}Q = \lambda\,\mathrm{d}x$.

The distance from this element to point P is given by $\left(a + l - x\right)$, so the field at P due to the element is in the x-direction, with a magnitude

$$\mathrm{d}E = \frac{\mathrm{d}q}{4\pi\varepsilon_0\left(a+l-x\right)^2} = \frac{\lambda \mathrm{d}x}{4\pi\varepsilon_0\left(a+l-x\right)^2}.$$

Since the field due to all other elements lies in the same direction, the total field at P is given by integrating over the length of the rod:

$$E=\int_0^l \frac{\lambda \mathrm{d}x}{4\pi\varepsilon_0 (a+l-x)^2}=\frac{\lambda}{4\pi\varepsilon_0}\left[\frac{1}{a+l-x}\right]_0^l$$
$$=\frac{\lambda}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{a+l}\right)=\frac{\lambda}{4\pi\varepsilon_0}\frac{a+l-a}{a(a+l)}=\frac{\lambda l}{4\pi\varepsilon_0 a(a+l)}$$

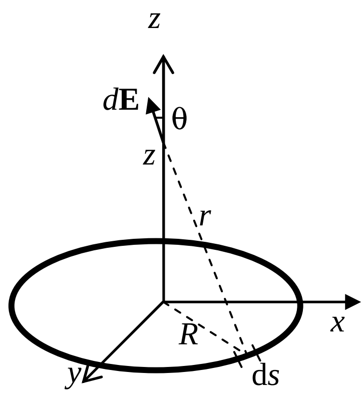
Exploiting the fact that $\lambda l = Q$, this can then be written as

$$\boxed{E=\frac{Q}{4\pi\varepsilon_0 a(a+l)}}$$

[We can check that this answer is reasonable as follows. If $a \gg l$, the rod will behave as a point charge. The expression above then reduces to $E \approx \frac{Q}{4\pi\varepsilon_0 a^2}$ as expected.]

Question 3

A charge Q is uniformly distributed along the circumference of a thin ring of radius R. What is the electric field at points along the axis of the ring?



Solution

Consider the ring to be made up of infinitesimal line segments ds as shown. Each of these, distributed around the ring, will contribute to the field at the point on the axis. We can use symmetry to determine the direction of the resultant field; it must be parallel to the z-axis, as the horizontal component due to any point on the ring will be exactly balanced by that due to the point diametrically opposite on the ring.

The charge per unit length along the circumference is $Q/2\pi R$; hence the charge on the line segment ds is $\mathrm{d}Q=\mathrm{d}s\frac{Q}{2\pi R}$. At a height z above the plane of the ring, the electric field due to this element of charge has magnitude

$$\mathrm{d}E=\frac{\mathrm{d}Q}{4\pi\varepsilon_0 r^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q\,\mathrm{d}s}{2\pi R}\frac{1}{z^2+R^2}\,.$$

The z-component of this is

$$\mathrm{d}E_z=\frac{1}{4\pi\varepsilon_0}\frac{Q\,\mathrm{d}s}{2\pi R}\frac{\cos\theta}{z^2+R^2}=\frac{1}{4\pi\varepsilon_0}\frac{Q\,\mathrm{d}s}{2\pi R}\frac{z}{\left(z^2+R^2\right)^{3/2}}$$

since $\cos\theta = z\big/\sqrt{z^2+R^2}$.

The overall electric field is therefore

$$E_z=\int\frac{1}{4\pi\varepsilon_0}\frac{Q}{2\pi R}\frac{z}{\left(z^2+R^2\right)^{3/2}}\mathrm{d}s\,.$$

The integrand has a constant value for all points around the ring, and so can be taken outside the integral, and $\int \mathrm{d}s$ is just the length of the circumference, $2\pi R$.

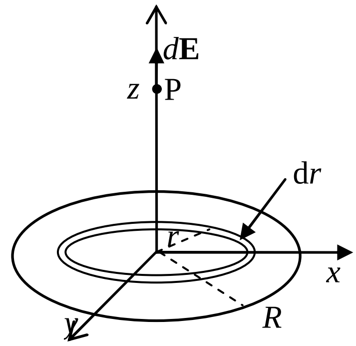
We therefore arrive at a resultant field

$$\boxed{\mathbf{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{\left(z^2+R^2\right)^{3/2}}\mathbf{k}}$$

[Once again, we can perform a couple of checks. For z = 0, $\mathbf{E} = 0$*, as it clearly must do by symmetry. For* $z \gg R$, $\mathbf{E} \approx \frac{Q}{4\pi\varepsilon_0 z^2}\mathbf{k}$ *as would be expected when the distribution can be treated as a point charge.]*

Question 4

What is the electric field generated by a disc of radius R carrying a uniform charge density σ C m^{-2}, at a point a distance z from the disc and on the axis of the disc?



Solution

Consider the disc to be made up of concentric rings, of thickness dr, one of which is shown. The charge dQ on this ring is equal to σ × its area, or $2\pi r\sigma\,\mathrm{d}r$. As proved in the previous question, the contribution to the field at P due to one ring of radius r is

$$\mathrm{d}E = \frac{\mathrm{d}Q\,z}{4\pi\varepsilon_0\left(z^2+r^2\right)^{3/2}} = \frac{r\sigma z\,\mathrm{d}r}{2\varepsilon_0\left(z^2+r^2\right)^{3/2}}\,.$$

The total field

$$E = \int_0^R \frac{r\sigma z\,\mathrm{d}r}{2\varepsilon_0\left(z^2+r^2\right)^{3/2}}\,.$$

This is easiest done by making the substitution $r^2 = u \quad \Rightarrow \quad 2r\,\mathrm{d}r = \mathrm{d}u$

$$E = \int_0^R \frac{r\sigma z\,\mathrm{d}r}{2\varepsilon_0\left(z^2+r^2\right)^{3/2}} = \int_0^{u_{\max}} \frac{\sigma z\,\mathrm{d}u}{4\varepsilon_0\left(z^2+u\right)^{3/2}} \qquad \text{where} \quad u_{\max} = R^2$$

$$= \frac{\sigma z}{4\varepsilon_0}\left[-\frac{2}{\left(z^2+u\right)^{1/2}}\right]_0^{u_{\max}} = \frac{\sigma z}{2\varepsilon_0}\left(-\frac{1}{\left(z^2+R^2\right)^{1/2}}+\frac{1}{z}\right)$$

$$= \frac{\sigma}{2\varepsilon_0}\left(1-\frac{z}{\left(z^2+R^2\right)^{1/2}}\right) = \frac{\sigma}{2\varepsilon_0}\left(1-\frac{1}{\left(1+\frac{R^2}{z^2}\right)^{1/2}}\right) \qquad \text{[Qu4]}$$

Question 5
What is the electric field generated by a large sheet carrying a uniform charge density σ C m^{-2}?

Solution
We can make use of the solution to the previous question by considering the field due to a disc in the limit that its radius R goes to infinity. In that case, the above expression [Qu4] reduces to

$$\boxed{E = \frac{\sigma}{2\varepsilon_0}}.$$

The electric field is therefore constant, and no longer depends on z. (This expression for E will appear later when we look at capacitors.)