polytropes

A full solution of the equations of stellar structure requires a complex computer code. Fortunately, by adopting various simplifying assumptions, it is possible to solve the equations of stellar structure using a much less complicated code, or even in some cases do without a computer altogether and solve the equations analytically. For example, if we adopt a simple relation between pressure and density which is valid throughout the star, the equations of hydrostatic support and mass conservation can be solved independently of the other 4 equations of stellar structure, resulting in a model in which the structure of the star is independent of the heat flowing through it. Such *polytropic models* played an important role in the development of stellar structure theory, particularly before the advent of powerful computers, and we will now look at them in detail.

Let us proceed by multiplying the equation of hydrostatic support by r^2 / ρ and differentiating with respect to r, giving

$$\mathrm{d} / \mathrm{d}r \, [(r^2 / \rho) \, \mathrm{d}P / \mathrm{d}r] = - G \, \mathrm{d}M / \mathrm{d}r \, .$$

Substituting the equation of mass conservation on the right-hand side, we obtain

$$(1 / r^2) \, \mathrm{d} / \mathrm{d}r \, [(r^2 / \rho) \, \mathrm{d}P / \mathrm{d}r] = - 4 \pi G \rho.$$

Let us now adopt an equation of state of the form

$$P = K \rho^{(n+1)/n},$$

where K is a constant and n is known as the *polytropic index*. Combining the last two equations, we obtain the following non-linear second-order differential equation for the density inside the star

$$(1 / r^2) \, \mathrm{d} / \mathrm{d}r \, [(r^2 / \rho) \, . \, [(n+1)K / n] \, . \, \rho^{1/n} \, \mathrm{d}\rho / \mathrm{d}r] = - 4 \pi G \rho,$$

which can be rearranged to

$$[(n+1)K / 4 \pi G n] \, . \, (1 / r^2) \, \mathrm{d} / \mathrm{d}r \, [(r^2 / \rho^{(n-1)/n}) \, \mathrm{d}\rho / \mathrm{d}r] = - \rho.$$

It is convenient at this point to replace r and ρ in the above equation by dimensionless variables. We therefore define a dimensionless variable, ξ, where

$$\xi = r / \alpha,$$

and α is a constant scale factor. We also define a dimensionless variable, θ, where

$$\theta^n = \rho / \rho_c$$

and ρ_c is the central density. Substituting these dimensionless variables into the second-order differential equation derived above, we obtain

$$[(n+1)K / 4 \pi Gn] \, . \, (1 / \alpha^2\xi^2) \, d / d\alpha\xi \, [(\alpha^2\xi^2 / \rho_c^{(n-1)/n} \, \theta^{n-1}) \, d\rho_c\theta^n / d\alpha\xi] = - \rho_c\theta^n,$$

which simplifies to

$$[(n+1)K / 4 \pi G \rho_c^{(n-1)/n}] \, . \, (1 / \alpha^2\xi^2) \, d / d\xi \, (\xi^2 \, d\theta / d\xi) = - \theta^n.$$

The coefficient in square brackets on the left-hand side of this equation is a constant having the dimension length squared. The square root of this coefficient is hence an appropriate choice for α, i.e.

$$\alpha = [(n+1)K / 4 \pi G \rho_c^{(n-1)/n}]^{0.5}.$$

Combining the last two equations, we obtain the *Lane-Emden equation*:

$$\mathbf{(1 / \xi^2) \, d / d\xi \, (\xi^2 \, d\theta / d\xi) = - \theta^n}.$$

To solve the Lane-Emden equation, we need two boundary conditions. At the centre of the polytrope, i.e. when $\xi = 0$, $\rho = \rho_c$ and hence $\theta = 1$. A second central boundary condition follows from the equation of hydrostatic support, in which the M/r^2 term tends to zero as r tends to zero. This means that $dP / dr = 0$ at $r=0$ and, from the polytropic equation of state, $d\theta / d\xi = 0$ at $\xi = 0$.

In the next section we will look at how these boundary conditions are used to solve the Lane-Emden equation and how the solutions compare to more sophisticated models of stellar structure.