accuracy of hydrostatic assumption

In deriving the equation of hydrostatic support, it has been assumed that the gravity and pressure forces are balanced in a star. How valid an assumption is this?

Let us consider again the element of mass in figure 6. We have seen that the outward force acting on the element is given by

$$P_r \, \delta S$$

and the inward force acting on the element is given by

$$P_{r+\delta r} \, \delta S + (GM_r / r^2) \, \rho_r \, \delta S \, \delta r.$$

If the inward and outward forces are not equal, there will be a resultant force acting on the element which will give rise to an acceleration, a. This resultant force is given by $\rho_r \, \delta S \, \delta r \, a$, where we have simply multiplied the acceleration of the element by its mass. Hence we can write

$$P_{r+\delta r} \, \delta S + (GM_r / r^2) \, \rho_r \, \delta S \, \delta r - P_r \, \delta S = \rho_r \, \delta S \, \delta r \, a,$$

where we have assumed an inward-acting resultant force. If we are considering an infinitesimal element, we can write,

$$P_{r+\delta r} = P_r + (\mathrm{d}P_r / \mathrm{d}r) \, \delta r \qquad \text{(in the limit } \delta r \to 0,\ \delta P_r / \delta r = \mathrm{d}P_r / \mathrm{d}r, \text{ where } \delta P_r = P_{r+\delta r} - P_r \text{)}.$$

Combining these last two equations and rearranging, we obtain:

$$\mathrm{d}P_r / \mathrm{d}r + (GM_r / r^2) \, \rho_r = \rho_r \, a.$$

Recalling that the acceleration due to gravity, g, is given by GM_r / r^2, and, for the sake of brevity, dropping the subscript r, we may write:

$$\mathrm{d}P / \mathrm{d}r + \rho g = \rho a.$$

This is a generalised form of the equation of hydrostatic support. We are now in a position to determine what happens if there is a resultant force acting on the element, i.e. if the sum of the two terms on the left-hand side of the above equation is not zero. Suppose that their sum is a small fraction β of the gravitational term, i.e.

$$\beta \rho g = \rho a.$$

This means that there will be an inward acceleration given by

$$a = \beta g.$$

If the element starts from rest with this acceleration, its inward displacement s after a time t will be given by:

$$s = \tfrac{1}{2}at^2 = \tfrac{1}{2}\beta g t^2.$$

Rewriting the above equation in terms of t, we obtain:

$$t = (1 / \beta^{1/2}) \times (2s / g)^{1/2}.$$

If we allow the element to fall all the way to the centre of the star, we can replace s in the above equation by r and then subsitute $g = GM / r^2$, giving:

$$t = (1 / \beta^{1/2}) \times (2r^3 / GM)^{1/2}.$$

The last term is known as the <u>*dynamical timescale*</u>, t_d , i.e.

$$t_d = (2r^3 / GM)^{1/2},$$

so we can write:

$$t = t_d / \beta^{1/2}.$$

This equation gives the time it would take a star to collapse if the forces are out of balance by a factor β.

Fossil and geological records indicate that the properties of the Sun have not changed significantly for at least 10^9 years (3×10^{16} s) and we know that the dynamical timescale for the Sun is approximately 2000 s (calculated by substituting values for the mass and radius of the Sun in the above expression for t_d). Hence, in the case of the Sun, we find that β can be no greater than 10^{-27}.

Not all stars are like the Sun, of course. During their lives, all stars undergo periods of radial expansion and/or contraction and at these times β will be much greater than this value. In such cases, the generalised form of the equation of hydrostatic support derived above must be used. Nevertheless, most stars *are* like the Sun (i.e. on the main sequence) and so we may conclude that:

the equation of hydrostatic support must be true to a very high degree of accuracy.

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