minimum mean temperature of a star

We are now in a position to determine the minimum mean temperature of a star. Consider the two terms in the virial theorem:

$$3\int_0^{V_s} P\,\mathrm{d}V + \Omega = 0.$$

We have seen that the gravitational potential energy term, Ω, can be written as:

$$-\Omega = \int_0^{M_s} (GM / r)\,\mathrm{d}M.$$

We can obtain an underestimate of this integral by noting that at all points inside the star $r < r_s$ and hence $1/r > 1/r_s$. This means that

$$\int_0^{M_s} (GM / r)\,\mathrm{d}M \quad > \int_0^{M_s} (GM / r_s)\,\mathrm{d}M = GM_s^2 / 2r_s.$$

Now, noting that $\mathrm{d}M = \rho\,\mathrm{d}V$, the virial theorem can be rewritten as follows:

$$-\Omega = 3\int_0^{V_s} P\,\mathrm{d}V = 3\int_0^{M_s} (P / \rho)\,\mathrm{d}M.$$

The pressure, P, in this equation is given by

$$P = P_{gas} + P_{rad}\ ,$$

where P_{gas} is the gas pressure and P_{rad} the radiation pressure. If we assume for the moment that stars are composed of an ideal gas with negligible radiation pressure, we can write:

$$P = nkT = kT\rho / m,$$

where n is the number of particles per cubic metre, m is the average mass of the particles in the stellar material and k is Boltzmann's constant. This gives for the virial theorem:

$$-\Omega = 3\int_0^{M_s} (P / \rho)\,\mathrm{d}M = 3\int_0^{M_s} (kT / m)\,\mathrm{d}M.$$

We may now rewrite the inequality derived above as follows:

$$- \Omega = 3 \int_0^{M_s} (kT / m) \, dM \; > \; GM_s^2 / 2r_s.$$

Rearranging gives:

$$\int_0^{M_s} T \, dM \; > \; GM_s^2 m / 6kr_s.$$

We can think of the integral on the left-hand side of the above equation as the sum of the temperatures of all of the infinitesimal mass elements dM which make up the star. The mean temperature of the star, T_{av}, is then given by this integral divided by the total mass of the star, M_s, i.e.

$$M_s T_{av} = \int_0^{M_s} T \, dM,$$

Combining the last two equations, we obtain an inequality which gives the minimum mean temperature of a star:

$$T_{av} \; > \; GM_s m / 6kr_s.$$

We will now use the above inequality to determine the state of stellar material.