Lecture 5 Polytropic models

The equation of hydrostatic equilibrium, discussed in the previous lecture, can be solved in the case where density ρ is a known function of pressure P. A particular example of this is a relation of the form

$$P = K\rho^{\gamma},$$

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(5.1)

where K and γ are constants; this is called a *polytropic relation*, and the resulting models are called *polytropic models*.

Models of this nature have played a very important role in the development of the subject; they are still very useful as simple examples which are, nevertheless, not too dissimilar from realistic models. More importantly, there are cases where the polytropic relation (5.1) is a very good approximation to reality. An example is where pressure and density are related adiabatically, as in equation (3.34).

To obtain the equation satisfied by polytropic models, we note that from equations (4.5) and (4.7) we have

$$\frac{d}{dr}\left(\frac{r^2}{\rho}\frac{dP}{dr}\right) = -G\frac{dm}{dr} = -4\pi G\rho r^2.$$

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(5.2)

Hence, using equation (5.1), we obtain

$$\frac{d}{dr}\left(r^2K\gamma\rho^{\gamma-2}\frac{d\rho}{dr}\right) = -4\pi G\rho r^2.$$

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(5.3)

It is convenient to replace γ by the polytropic index n, defined by

$$n = \frac{1}{\gamma - 1} \quad , \quad \gamma = 1 + \frac{1}{n}.$$

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(5.4)

Furthermore, we introduce a dimentionless measure θ of density ρ by

$$\rho = \rho_c \theta^n,$$

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(5.5)

where ρ_c is central density. Then equation (5.3) becomes

$$\frac{d}{dr}\left(r^2 K \frac{n+1}{n} \rho_c^{1/n-1} \theta^{1-n} \rho_c n \theta^{n-1} \frac{d\theta}{dr}\right) = -4\pi G \rho_c \theta^n r^2$$

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or

$$\frac{(n+1) K \rho_c^{1/n-1}}{4\pi G} \frac{1}{r^2} \frac{d}{dr}\left(r^2 \frac{d\theta}{dr}\right) = -\theta^n.$$

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(5.6)

To simplify the equation further, we introduce a new measure ξ for the distance to the centre by

$$r = \alpha \xi \quad , \text{where} \quad \alpha^2 = \frac{(n+1) K \rho_c^{1/n-1}}{4\pi G}.$$

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(5.7)

Then the equation finally becomes

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n.$$

(5.8)

This equation is called the *Lane-Emden equation*, and the solution $\theta = \theta(\xi)$ is called the *Lane-Emden function*. From equation (5.5) it follows that θ must satisfy the boundary condition

$$\theta(\xi) = 1 \quad \text{for} \quad \xi = 0.$$

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(5.9)

The surface of the model is defined by the point $\xi = \xi_1$, where $\theta(\xi_1) = 0$.

Given the solution $\theta(\xi)$, we can obtain relations between the various quantities characterizing the model. It follows immediately from equation (5.7) that the surface radius of the model is

$$R = \alpha\xi_1 = \left[\frac{(n+1)K\rho_c^{1/n-1}}{4\pi G}\right]^{1/2}\xi_1.$$

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(5.10)

The mass $m(\xi)$ interior to ξ may be obtained by integrating equation (4.7), using equations (5.5, 5.7, 5.8) as

$$m(\xi) = \int_0^{\alpha\xi} 4\pi r^2 \rho \, dr = 4\pi\alpha^3\rho_c \int_0^{\xi} \xi^2 \theta^n d\xi$$
$$= -4\pi\alpha^3\rho_c \int_0^{\xi} \frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) d\xi$$
$$= -4\pi\alpha^3\rho_c \xi^2 \frac{d\theta}{d\xi} .$$

$$m(\xi) = \int_0^{\alpha\xi} 4\pi r^2 \rho \, dr = 4\pi\alpha^3\rho_c \int_0^{\xi} \xi^2 \theta^n d\xi$$
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$$= -4\pi\alpha^3\rho_c \xi^2 \frac{d\theta}{d\xi} .$$

(5.11)

Using the expression (5.7) for α, we finally obtain

$$m(\xi) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \xi^2 \frac{d\theta}{d\xi} .$$

$$m(\xi) = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \xi^2 \frac{d\theta}{d\xi} .$$ (5.12)

In particular, the total mass is given by

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} .$$

$$M = -4\pi\left[\frac{(n+1)K}{4\pi G}\right]^{3/2} \rho_c^{\frac{3-n}{2n}} \left(\xi^2 \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} . \qquad (5.13)$$

From equations (5.10) and (5.13), by eliminating ρ_c, we may find a relation between M, R and K. The result is

$$K = (4\pi)^{\frac{1}{n}} \frac{G}{n+1} \xi_1^{-\frac{n+1}{n}} \left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}^{\frac{1-n}{n}} M^{\frac{n-1}{n}} R^{\frac{3-n}{n}} .$$

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There are two different interpretations of this relation. If the constant K in equation (5.1) is given in terms of basic physical constants and hence is known, equation (5.14) defines a relation between the mass and the radius of the star. If, on the other hand, equation (5.1) just expresses proportionality, the constant K being essentially arbitrary, then equation (5.14) may be used to determine K for a star with a given mass and radius; as shown below one may then determine other quantities for the star. In the former case, therefore, there is a unique polytropic model for a given mass, whereas in the latter case a model can be constructed for any value of M and R.

Exercise 5.1. Verify equation (5.14). 

From the last of equations (5.11) we find that the mean density of the star is

$$\bar{\rho} = \frac{3M}{4\pi R^3} = -\frac{3}{\xi_1}\left(\frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \rho_c,$$

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(5.15)

and hence the central density is determined by the mass and radius as

$$\rho_c = -\left(\frac{\xi}{3} / \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \frac{3M}{4\pi R^3} \equiv a_n \frac{3M}{4\pi R^3},$$

$$\rho_c = -\left(\frac{\xi}{3} / \frac{d\theta}{d\xi}\right)_{\xi=\xi_1} \frac{3M}{4\pi R^3} \equiv a_n \frac{3M}{4\pi R^3},$$ (5.16)

where the last equation defines constant a_n which depends on the polytropic index n only. Finally, using that from equation (5.1)

$$P_c = K\rho_c^{(n+1)/n},$$ $$P_c = K\rho_c^{(n+1)/n},$$

(5.17)

and using equations (5.14) and (5.16), we find that

$$P_c = \frac{1}{4\pi(n+1)}\left(-\frac{d\theta}{d\xi}\right)^{-2}_{\xi=\xi 1} \frac{GM^2}{R^4} \equiv c_n \frac{GM^2}{R^4},$$

$$P_c = \frac{1}{4\pi(n+1)}\left(-\frac{d\theta}{d\xi}\right)^{-2}_{\xi=\xi 1} \frac{GM^2}{R^4} \equiv c_n \frac{GM^2}{R^4},$$ (5.18)

where c_n depends on the polytropic index n only. The pressure throughout the model is then determined by

$$P = P_c\theta^{n+1}. \tag{5.19}$$

$$P = P_c\theta^{n+1}.$$

Exercise 5.2. Fill in the missing details in the derivation of equations (5.15), (5.16), and (5.18). 

In the case where the temperature is related to pressure and density through the ideal gas law (3.13), it may be determined from equations (5.5) and (5.19) as

$$T = T_c\theta, \tag{5.20}$$

$$T = T_c\theta,$$

where

$$T_c = \left[(n+1)\xi_1\left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{-1}\frac{GM\mu m_H}{kR} \equiv b_n\frac{GM\mu m_H}{kR}, \tag{5.21}$$

$$T_c = \left[(n+1)\xi_1\left(-\frac{d\theta}{d\xi}\right)_{\xi=\xi_1}\right]^{-1}\frac{GM\mu m_H}{kR} \equiv b_n\frac{GM\mu m_H}{kR},$$

where b_n depends on the polytropic index n only. In the case when the star is composed of an ideal gas, therefore, θ is a measure of the temperature.

To determine the structure of a polytropic star completely, we only need to find the solution to the Lane-Emden equation (5.8). Unfortunately, in general no analytical solution is possible. The only exceptions are $n=0$, 1 and 5 where the solutions are

$$n = 0: \qquad \theta = 1 - \xi^2/6, \qquad \xi_1 = \sqrt{6}$$

$$n = 0: \qquad \theta = 1 - \xi^2/6, \qquad \xi_1 = \sqrt{6}$$
(5.22)

$$n = 1: \qquad \theta = \frac{\sin \xi}{\xi}, \qquad \xi_1 = \pi$$

$$n = 1: \qquad \theta = \frac{\sin \xi}{\xi}, \qquad \xi_1 = \pi$$

(5.23)

$$n = 5: \qquad \theta = \left(1 + \xi^2/3\right)^{-1/2}, \qquad \xi_1 = \infty.$$

$$n = 5: \qquad \theta = \left(1 + \xi^2/3\right)^{-1/2}, \qquad \xi_1 = \infty.$$
(5.24)

Exercise 5.3. Verify that these solutions satisfy the Lane-Emden equation (5.8) and the boundary condition (5.9).

The solution for $n=5$ is evidently peculiar, in that it has infinite radius. On the other hand, since

$$\lim_{\xi \to \infty} \left(-\xi^2 \frac{d\theta}{d\xi} \right) = \sqrt{3}, \qquad n = 5$$

$$\lim_{\xi \to \infty} \left(-\xi^2 \frac{d\theta}{d\xi} \right) = \sqrt{3}, \qquad n = 5$$

(5.25)

is finite, so is the mass of the model. It may be shown that only for $n<5$ does the Lane-Emden equation have solutions corresponding to finite radius.

For values of n other than 0 and 1 , the Lane-Emden equation must be solved numerically. Extensive tables of the solution exist; in any case, with modern computational facilities the solution of the equation is a simple numerical problem. Table 5.1 lists a number of useful quantities, which enter into the expressions given above, for a selection of polytropic models.

n	ξ_1	a_n	b_n	c_n
0	2.449	1.00	0.5	0.12
1	3.142	3.29	0.5	0.39
1.5	3.654	5.99	0.54	0.77
2	4.353	11.40	0.60	1.64
3	6.897	54.18	0.85	11.05
4	14.97	662.4	1.67	247.6

Table 5.1. Properties of polytropic models. Constants a_n, b_n and c_n specify the central density, central temperature and central pressure as given by equations (5.16), (5.18) and (5.21).

The next table, Table 5.2, presents the solution for two particular cases $n=1.5$ and $n=3$, at selected values of ξ.

$n = 1.5$ $n = 3$

$n = 1.5$ $n = 3$

ξ	θ	$d\theta/d\xi$		ξ	θ	$d\theta/d\xi$
0	1	0		0	1	0
0.5	0.96	-0.16		0.5	0.96	-0.16
1.0	0.85	-0.29		1.0	0.86	-0.25
1.5	0.68	-0.36		1.5	0.72	-0.28
2.0	0.50	-0.37		2.0	0.58	-0.26
2.5	0.32	-0.34		3.0	0.36	-0.18
3.0	0.16	-0.28		4.0	0.21	-0.12
3.5	0.03	-0.22		6.0	0.04	-0.06

3.654	0	-0.22		6.897	0	-0.04

Table 5.2. Properties of polytropes of indices $n=1.5$ and $n=3$.

From Table 5.1 it follows that the properties of polytropic models vary widely with n. This is true in particular of the degree of central condensation, as measured by a_n, the ratio between central and mean density. For $n=0$ it is obvious from equation (5.5) that density ρ is constant, and hence $a_1=1$, whereas the value of a_n tends to infinity as $n \to 5$. For stars on the main sequence the central condensation is typically around 10^2, corresponding to a polytrope of index around 3.3.

It should be noticed also that equation (5.18) for the central pressure and, in the ideal gas case, equation (5.21) for the central temperature, confirm the simple scaling derived in the previous Lecture (section 4.2). Now, however, the polytropic relations contain the additional numerical constants b_n and c_n . It is obvious from Table 5.1 that that c_n varies strongly with n; hence the estimate in equation (4.9) of the central pressure is at most a rough approximation. On the other hand, the range of variation of b_n is much more modest, except when n is very close to the critical case $n=5$. Thus equation (4.10) gives a reasonable estimate for the central temperature for a wide range of models.

Exercise 5.4. Find ρ_c , P_c and T_c in a polytrope of index 3 with solar mass (2.00×10^{30}kg) and radius (6.96×10^8m) and chemical composition $X=0.7$, $Z=0.02$, where the ideal gas equation of state is assumed to be valid. Find also ρ , P and T at the point where $r=R/2$. (Use the data in Tables 5.1 and 5.2).