

## Lecture 3

### Equation of state: the ideal gas

The description of stellar structure requires knowledge about properties of the matter in the stars. The properties of the gas are described in terms of local state variables, and the relations between them. For example, given the values of the temperature  $T$ , the density  $\rho$  and the chemical composition, it is possible to calculate all other state variables, including the pressure  $p$  and the internal energy per unit volume  $U$ . The specification of these relations constitutes the definition of the *equation of state* satisfied by the gas.

A considerable simplification results from the high temperature in stellar interiors, which means that in most of the star the gas can be assumed to be fully ionized, i.e. decomposed into bare nuclei and free electrons. Therefore the gas consists of particles with no internal degrees of freedom. Also, as a first approximation one can neglect the interactions between the particles. Such a gas is described as being an *ideal gas*.

#### 3.1. The ideal gas

The distribution of the speed  $v$  in a gas consisting of "classical" particles is given by the *Maxwell distribution*:

$$f(v) = 4\pi \left( \frac{m}{2\pi kT} \right)^{3/2} \exp\left( -\frac{mv^2}{2kT} \right) v^2,$$

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where  $m$  is the mass of the particle and  $k$  is Boltzmann's constant. This distribution function is defined such that  $f(v)dv$  gives the probability of finding the particle in the range of speed between  $v$  and  $v+dv$ . Accordingly  $f$  is normalized such that

$$\int_0^{\infty} f(v) dv = 1. \quad \int_0^{\infty} f(v) dv = 1. \quad (3.2)$$

Using the Maxwell distribution, we can calculate average value of the kinetic energy  $mv^2/2$  of an individual particle:

$$\left\langle \frac{mv^2}{2} \right\rangle = \int_0^{\infty} \frac{mv^2}{2} f(v) dv = \frac{3}{2} kT,$$

$$\left\langle \frac{mv^2}{2} \right\rangle = \int_0^{\infty} \frac{mv^2}{2} f(v) dv = \frac{3}{2} kT,$$

(3.3)

where we use  $\langle \rangle$  to designate the average value of corresponding physical quantity.

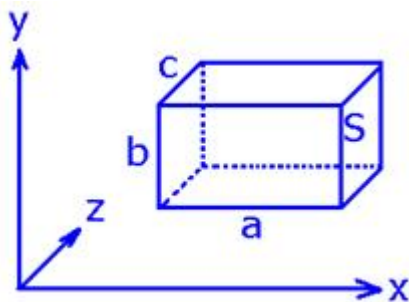
Exercise 3.1. Verify equations (3.2, 3.3). You can assume without prove that

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}.$$

$$\int_{-\infty}^{\infty} \exp(-x^2) dx = \sqrt{\pi}. \quad (3.4)$$



Consider now the ideal gas of temperature  $T$  contained in a rectangular box of dimensions  $a$ ,  $b$  and  $c$  as shown below:



When a single particle of mass  $m$  and velocity  $\mathbf{v} = (v_x, v_y, v_z)$  hits the wall of the box labeled with area  $S = bc$ , the particle changes its momentum by an amount  $\Delta(mv_x) = 2mv_x$ . Momentum conservation ( $F\Delta t = \Delta(mv)$  where  $F$  is force and  $\Delta t$  is time interval over which the force is applied) requires this momentum transfer to be balanced by the external force  $F_1$  applied to the wall from the outside:

$$\langle F_1 \rangle \Delta t = 2mv_x. \quad (3.5)$$

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For the time interval, we shall use  $\Delta t = 2a/v_x$ , the time interval between two consecutive hits. Notation  $\langle \rangle$  is used here to designate the time average of the force  $F$ , and subscript 1 is to remind that we are dealing with just one particle. This force is produced by the external pressure  $P_1$ ; we have  $\langle F_1 \rangle = P_1 S$ , and thus

$$P_1 V = mv_x^2, \quad P_1 V = mv_x^2, \quad (3.6)$$

where  $V = aS = abc$  is the box volume.

Let  $N$  be the total number of particles in volume  $V$ . Each of the particles contributes to the pressure; for the total pressure  $P$  we get

$$PV = N \langle mv_x^2 \rangle, \quad PV = N \langle mv_x^2 \rangle, \quad (3.7)$$

where  $\langle \rangle$  now designates averaging over the particles.

We also have

$$\left\langle \frac{mv_x^2}{2} \right\rangle = \left\langle \frac{mv_y^2}{2} \right\rangle = \left\langle \frac{mv_z^2}{2} \right\rangle, \\ \left\langle \frac{mv_x^2}{2} \right\rangle = \left\langle \frac{mv_y^2}{2} \right\rangle = \left\langle \frac{mv_z^2}{2} \right\rangle, \quad (3.8)$$

i.e the kinetic energy equipartition between three degrees of freedom, and

$$\left\langle \frac{mv_x^2}{2} \right\rangle + \left\langle \frac{mv_y^2}{2} \right\rangle + \left\langle \frac{mv_z^2}{2} \right\rangle = \left\langle \frac{mv^2}{2} \right\rangle.$$

$$\left\langle \frac{mv_x^2}{2} \right\rangle + \left\langle \frac{mv_y^2}{2} \right\rangle + \left\langle \frac{mv_z^2}{2} \right\rangle = \left\langle \frac{mv^2}{2} \right\rangle. \quad (3.9)$$

Equation (3.3) thus gives  $\langle mv_x^2 \rangle = kT$ ; from the equation (3.7) we get finally

$$PV = NkT, \quad PV = NkT, \quad (3.10)$$

the equation of state of the ideal gas.

In the stellar case we are not concerned with a given volume of gas, but rather with its microscopic properties, and hence it is more convenient to write equation (3.10) as

$$P = nkT, \quad P = nkT, \quad (3.11)$$

where  $n=N/V$  is the number of particles per unit volume. Introducing the mass density  $\rho$  and the (dimensionless) atomic weight  $\mu$  of the particles, we have

$$n = \frac{\rho}{\mu m_H}, \quad n = \frac{\rho}{\mu m_H}, \quad (3.12)$$

where  $m_H$  is the mass of the hydrogen atom,  $m_H=1.67 \times 10^{-27}$  kg, and hence

$$P = \frac{\rho kT}{\mu m_H}. \quad P = \frac{\rho kT}{\mu m_H}. \quad (3.13)$$

The ratio  $k/m_H$  is known as the *universal gas constant*

$$\mathfrak{R} = \frac{k}{m_H} = 8.31 \cdot 10^3 \text{ JK}^{-1}\text{kg}^{-1},$$

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(3.14)

and another form of the equation of state of the ideal gas (3.14) is

$$P = \frac{\rho \mathfrak{R}T}{\mu}. \quad P = \frac{\rho \mathfrak{R}T}{\mu}. \quad (3.15)$$

### 3.2. Mean molecular weight

In practice, stellar matter consists of a mixture of different elements, the atoms of which are largely or fully ionized. Thus the description in the previous section, which assumed only one kind of particles, is too simple. However, it is straightforward to generalize it.

If the gas consists of different types of particles each behaving like an ideal gas, with number densities  $n_i$ , the total pressure in the gas is obtained as the sum of the partial pressures  $P_i$ :

$$P = \sum_i P_i = \sum_i n_i kT. \quad (3.16)$$

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Note that when having the same number densities, electrons make the same contribution to the pressure as nuclei, despite their smaller mass.

Let us consider a mixture of atoms of different elements, all of which are assumed to be fully ionized. We denote the atomic number of element  $i$  as  $Z_i$ , its atomic weight as  $A_i$ , and its mass fraction as  $X_i$ . When fully ionized, each atom contributes  $Z_i + 1$  particles ( $Z_i$  electrons and one nucleus).

The number of atoms of element  $i$  per unit volume is  $\rho X_i / (A_i m_H)$ , and hence the total number of particles per unit volume for element  $i$  is  $\rho X_i (Z_i + 1) / (A_i m_H)$ . Thus, from the equation (3.16) it follows that the pressure is

$$P = \sum_i \rho X_i \frac{Z_i + 1}{A_i m_H} kT \equiv \frac{\rho kT}{\mu m_H},$$

$$P = \sum_i \rho X_i \frac{Z_i + 1}{A_i m_H} kT \equiv \frac{\rho kT}{\mu m_H}, \quad (3.17)$$

where we introduced the *mean molecular weight*  $\mu$  by

$$\mu^{-1} = \sum_i X_i \frac{Z_i + 1}{A_i}.$$

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(3.18)

Therefore, with this definition of  $\mu$  we recover equation (3.13) for  $P$ .

It is conventional to denote the mass fraction of H and He by  $X$  and  $Y$ , respectively, and the mass fraction of the remaining, so-called heavy, elements by  $Z$ . This separation is useful because  $Z \ll X, Y$  in most stellar compositions. Note that the normalization  $X + Y + Z = 1$  must hold. To obtain an approximate expression for  $\mu$  we take  $A_1 = 1$  for hydrogen,  $A_2 = 4$  for helium, and approximate  $(Z_i + 1)/A_i$  by  $1/2$  for heavy elements. Then

$$\mu^{-1} \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z,$$

$$\mu^{-1} \approx 2X + \frac{3}{4}Y + \frac{1}{2}Z, \quad (3.19)$$

or

$$\mu \approx \frac{4}{3 + 5X - Z}.$$

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(3.20)

### 3.3. Specific heats and adiabatic change

The internal energy of the monoatomic ideal gas is just the kinetic energy of the thermal motion of its particles. The mean internal energy per particle is  $3/2kT$ , as given by equation (3.3). Thus the internal energy per unit volume is

$$u = \frac{3}{2}nkT = \frac{3}{2} \frac{\rho kT}{\mu m_H} = \frac{3}{2}P.$$

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The energetics of the gas as the star evolves, or the gas moves, plays a very important role for stellar evolution. The basic equation describing the changes in the properties of the gas is the *first law of thermodynamics*,

$$dQ = dU + PdV, \quad dQ = dU + PdV,$$

(3.22)

valid for a fixed amount of matter, where  $U$  is the internal energy of the matter, and  $V$  is the volume it occupies. Here  $dQ$  is an amount of heat added to the matter, which, as expressed in the right-hand side, goes partially into changing the internal energy, and partially into work done against the external pressure, to change the volume of the gas. We let  $V$  to be the volume corresponding to unit mass, so that

$$V = \frac{1}{\rho}. \quad V = \frac{1}{\rho}.$$

(3.23)

Then  $U = u/\rho$  is the *specific internal energy*, i.e. the internal energy per unit mass; from equation (3.21) it follows that

$$U = \frac{3}{2} \frac{kT}{\mu m_H}. \quad U = \frac{3}{2} \frac{kT}{\mu m_H}. \quad (3.24)$$

We now consider a process where the volume does not change. We introduce the *specific heat per unit volume*,  $c_V$ , as an amount of heat that has to be added, per unit mass, to raise the temperature one degree. It follows from equations (3.22) and (3.24) that

$$dQ = \frac{3}{2} \frac{k}{\mu m_H} dT, \quad dQ = \frac{3}{2} \frac{k}{\mu m_H} dT, \quad (3.25)$$

and hence

$$c_V = \frac{3}{2} \frac{k}{\mu m_H}. \quad c_V = \frac{3}{2} \frac{k}{\mu m_H}. \quad (3.26)$$

It is also of interest to consider a process where the pressure is constant. To do so, we use the ideal gas law of the form  $PV=NkT$  (equation 3.10), where in the present case (unit mass) we have  $N=1/(\mu m_H)$ . For the changes we then obtain

$$PdV + VdP = \frac{k}{\mu m_H} dT, \quad PdV + VdP = \frac{k}{\mu m_H} dT, \quad (3.27)$$

and hence from equation (3.22)

$$dQ = dU - VdP + \frac{k}{\mu m_H} dT$$

$$= \frac{5}{2} \frac{k}{\mu m_H} dT - Vdp$$

$$dQ = dU - VdP + \frac{k}{\mu m_H} dT$$

$$= \frac{5}{2} \frac{k}{\mu m_H} dT - Vdp$$

(3.28)

by using equation (3.24). From this equation it follows that the *specific heat at constant pressure* is

$$c_p = \frac{5}{2} \frac{k}{\mu m_H}.$$

$$c_p = \frac{5}{2} \frac{k}{\mu m_H}.$$

(3.29)

A particularly important type of process are the *adiabatic* processes, which occur without any exchange of heat, i.e. with  $dQ=0$ . For such a process, from equations (3.22), (3.24) and (3.26) we obtain

$$c_V dT = -PdV.$$

(3.30)

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To obtain a relation between the changes in  $P$  and  $V$  (or  $\rho$ ) for an adiabatic process, we use the ideal gas law in the form (3.27), which is equivalent to

$$\frac{dV}{V} + \frac{dP}{P} = \frac{dT}{T},$$

(3.31)

$$\frac{dV}{V} + \frac{dP}{P} = \frac{dT}{T},$$

to write the equation (3.30) as



$$\begin{aligned}
c_V \left( \frac{dP}{P} + \frac{dV}{V} \right) &= -\frac{P}{T} dV \\
&= -\frac{k}{\mu m_H} \frac{dV}{V} = (c_V - c_P) \frac{dV}{V} \\
c_V \left( \frac{dP}{P} + \frac{dV}{V} \right) &= -\frac{P}{T} dV \\
&= -\frac{k}{\mu m_H} \frac{dV}{V} = (c_V - c_P) \frac{dV}{V}
\end{aligned} \tag{3.32}$$

and hence


$$\begin{aligned}
\frac{dP}{P} = -\frac{c_P}{c_V} \frac{dV}{V} = -\gamma \frac{dV}{V} = \gamma \frac{d\rho}{\rho}, \\
\frac{dP}{P} = -\frac{c_P}{c_V} \frac{dV}{V} = -\gamma \frac{dV}{V} = \gamma \frac{d\rho}{\rho},
\end{aligned} \tag{3.33}$$

where we introduced  $\gamma \equiv c_P/c_V$  and used equation (3.23). For the ideal gas which we are considering it follows from the explicit expressions (3.26, 3.29) for  $c_V$  and  $c_P$  that  $\gamma = 5/3$ . We may also write equation (3.33) as

$$\left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S = \gamma, \quad \left( \frac{\partial \ln P}{\partial \ln \rho} \right)_S = \gamma, \tag{3.34}$$

where the subscript  $S$  indicates that the partial derivative is taken at constant *specific entropy*, i.e. without any heat exchange.

### Further exercises

**Exercise 3.2.** Show from first principles that for a star consisting of ionized hydrogen and ionized helium only, the mean molecular weight is  $\mu = 4/(3+5X)$ . Show that this is also obtained by putting  $Z=0$  in the general expression. 

**Exercise 3.3.** An evolved star consists of equal amounts of ionized helium and an element of atomic mass 14 with 6 electrons. Both elements are fully ionized. Obtain the exact value of the mean molecular weight  $\mu$ .

