Chapter 9

Systems of Particles

9.1 The Centre of Mass

Consider a system of n particles, with masses m_i and position vectors $\mathbf{x}_i$ $(i = 1, \dots, n)$. The force on the i^{th} particle consists of the "internal" forces from each of the other particles in the system — say $\mathbf{F}_{ij}$ from the j^{th} particle — plus the external force from outside the system, denoted by $\mathbf{F}_i^{\text{ext}}$. Note that

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

by $\mathscr{N}$ ***III***, so that

$$\boxed{\sum_{i=1}^{n} \sum_{j \neq i} \mathbf{F}_{ij} = \mathbf{0}} \tag{9.1}$$

(because for each pair (I, J) of particles, the term $\mathbf{F}_{IJ}$ which is in the sum cancels with the term $\mathbf{F}_{JI}$ which is also in the sum). Hence the total *internal* force is zero.

Define the total mass

$$M = \sum_{i=1}^{n} m_i,$$

the *centre of mass* (CoM)

$$\boxed{\mathbf{X} = \frac{1}{M} \sum_{i=1}^{n} m_i \mathbf{x}_i}$$

and the total external force

$$\mathbf{F} = \sum_{i=1}^{n} \mathbf{F}_i^{\text{ext}}.$$

Then by $\mathscr{N}$ ***II***,

$$m_i \ddot{\mathbf{x}}_i = \mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij}$$

$$\implies \qquad \sum_i m_i \ddot{\mathbf{x}}_i = \mathbf{F} + \sum_i \sum_{j \neq i} \mathbf{F}_{ij}$$

$$\implies \qquad M\ddot{\mathbf{X}} = \mathbf{F}.$$

So the centre of mass moves as if it were a particle of mass M acted on by the total external force $\mathbf{F}$. The total momentum of the system is

$$\mathbf{P} = \sum_{i=1}^{n} m_i \dot{\mathbf{x}}_i,$$

so

$$\boxed{\dot{\mathbf{P}} = M\ddot{\mathbf{X}} = \mathbf{F}.} \tag{9.2}$$

If the total external force were zero then the centre of mass would move with constant velocity; if we wished, we could therefore make a Galilean transformation to the centre of mass frame (in which the centre of mass is fixed at the origin, i.e., $\mathbf{X} = \mathbf{0}$). However, we shall not do so here.

Let

$$\mathbf{x}_i' = \mathbf{x}_i - \mathbf{X}$$

be the position vector of particle i relative to the centre of mass (i.e., $\mathbf{x}_i'$ is the position vector of particle i in the centre of mass frame). Then

$$\sum_i m_i \mathbf{x}_i' = \sum_i m_i \mathbf{x}_i - \sum_i m_i \mathbf{X} = M\mathbf{X} - M\mathbf{X}$$

$$\implies \qquad \boxed{\sum_{i=1}^{n} m_i \mathbf{x}_i' = \mathbf{0}.} \tag{9.3}$$

9.2 Angular Momentum

We shall *assume* here that the internal forces are all central forces, that is to say, $\mathbf{F}_{ij}$ acts along the line joining particles i and j. (This is true for gravitational, electrostatic and elastic forces, etc.) Then

$$\mathbf{F}_{ij} \parallel (\mathbf{x}_i - \mathbf{x}_j) \qquad \forall\, i, j.$$

Therefore

$$\boxed{\sum_{i=1}^{n} \sum_{j \neq i} \mathbf{x}_i \times \mathbf{F}_{ij} = \mathbf{0}} \tag{9.4}$$

since the contribution from any pair (I, J) of particles is

$$\mathbf{x}_I \times \mathbf{F}_{IJ} + \mathbf{x}_J \times \mathbf{F}_{JI} = (\mathbf{x}_I - \mathbf{x}_J) \times \mathbf{F}_{IJ} = \mathbf{0}.$$

Define the total external *couple* or *torque* about a point $\mathbf{a}(t)$ (not necessarily fixed, nor necessarily the centre of mass) by

$$\boxed{\mathbf{G} = \sum_{i=1}^{n} (\mathbf{x}_i - \mathbf{a}) \times \mathbf{F}_i^{\text{ext}}.}$$

The total angular momentum about $\mathbf{a}$ is given by

$$\boxed{\mathbf{H} = \sum_{i=1}^{n} m_i (\mathbf{x}_i - \mathbf{a}) \times (\dot{\mathbf{x}}_i - \dot{\mathbf{a}}).}$$

Hence

$$\begin{aligned}
\dot{\mathbf{H}} &= \sum_i m_i (\mathbf{x}_i - \mathbf{a}) \times (\ddot{\mathbf{x}}_i - \ddot{\mathbf{a}}) \\
&= \sum_i (\mathbf{x}_i - \mathbf{a}) \times \left(\mathbf{F}_i^{\text{ext}} + \sum_{j \neq i} \mathbf{F}_{ij} - m_i \ddot{\mathbf{a}} \right) \\
&= \mathbf{G} + \sum_i \sum_{j \neq i} \mathbf{x}_i \times \mathbf{F}_{ij} - \mathbf{a} \times \sum_i \sum_{j \neq i} \mathbf{F}_{ij} - \left\{ \sum_i (m_i \mathbf{x}_i - m_i \mathbf{a}) \right\} \times \ddot{\mathbf{a}} \\
&= \mathbf{G} - M(\mathbf{X} - \mathbf{a}) \times \ddot{\mathbf{a}}. && \text{(using (9.1) and (9.4))}
\end{aligned}$$

In particular, *if* $\mathbf{a}$ *is either the centre of mass or if it is fixed*, then

$$\boxed{\dot{\mathbf{H}} = \mathbf{G}.} \tag{9.5}$$

We shall always use these “safe” cases.

Example of a couple: suppose that the external force on each particle is just gravity (near the Earth’s surface), so $\mathbf{F}_i^{\text{ext}} = m_i \mathbf{g}$. Then

$$\begin{aligned}
\mathbf{G} &= \sum_i m_i (\mathbf{x}_i - \mathbf{a}) \times \mathbf{g} \\
&= (M\mathbf{X} - M\mathbf{a}) \times \mathbf{g} \\
&= (\mathbf{X} - \mathbf{a}) \times M\mathbf{g}.
\end{aligned}$$

As we might have expected, this is exactly the same as the couple that would be produced by a single particle of mass M at the centre of mass.

We can write

$$\begin{aligned}\mathbf{H} &= \sum_i m_i(\mathbf{X} - \mathbf{a} + \mathbf{x}'_i) \times (\dot{\mathbf{X}} - \dot{\mathbf{a}} + \dot{\mathbf{x}}'_i) \\ &= \sum_i m_i(\mathbf{X} - \mathbf{a}) \times (\dot{\mathbf{X}} - \dot{\mathbf{a}}) + \sum_i m_i \mathbf{x}'_i \times (\dot{\mathbf{X}} - \dot{\mathbf{a}}) \\ &\qquad\qquad + (\mathbf{X} - \mathbf{a}) \times \sum_i m_i \dot{\mathbf{x}}'_i + \sum_i m_i \mathbf{x}'_i \times \dot{\mathbf{x}}'_i;\end{aligned}$$

so, using (9.3) and its derivative,

$$\boxed{\mathbf{H} = M(\mathbf{X} - \mathbf{a}) \times (\dot{\mathbf{X}} - \dot{\mathbf{a}}) + \sum_{i=1}^{n} m_i \mathbf{x}'_i \times \dot{\mathbf{x}}'_i.}$$

Hence the total angular momentum about $\mathbf{a}$ is the “angular momentum of the centre of mass about $\mathbf{a}$” plus the angular momemtum about the centre of mass.

9.3 Energy of the System

It can be useful to write the total kinetic energy of the system as follows:

$$\begin{aligned}T &= \sum_i \tfrac{1}{2} m_i \dot{\mathbf{x}}_i^2 \\ &= \sum_i \tfrac{1}{2} m_i (\dot{\mathbf{a}} + \dot{\mathbf{x}}_i - \dot{\mathbf{a}})^2 \\ &= \sum_i \tfrac{1}{2} m_i \dot{\mathbf{a}}^2 + \sum_i m_i \dot{\mathbf{a}} \,.\, (\dot{\mathbf{x}}_i - \dot{\mathbf{a}}) + \sum_i \tfrac{1}{2} m_i (\dot{\mathbf{x}}_i - \dot{\mathbf{a}})^2 \\ &= \tfrac{1}{2} M \dot{\mathbf{a}}^2 + \dot{\mathbf{a}} \,.\, (M\dot{\mathbf{X}} - M\dot{\mathbf{a}}) + \sum_i \tfrac{1}{2} m_i (\dot{\mathbf{x}}_i - \dot{\mathbf{a}})^2.\end{aligned}$$

The last of these three terms is just the kinetic energy of the system *relative to* $\mathbf{a}$, which we shall denote T_{rel}. The second term vanishes, because we are assuming either $\mathbf{a} = \mathbf{X}$ or $\dot{\mathbf{a}} = \mathbf{0}$ according to the “safe” cases described in §9.2. Hence

$$\boxed{T = \tfrac{1}{2} M \dot{\mathbf{a}}^2 + T_{\text{rel}}.}$$

In particular, if $\mathbf{a} = \mathbf{X}$, we see that the total kinetic energy is the “kinetic energy of the centre of mass” plus the “kinetic energy relative to the centre of mass”.

Suppose that all the internal and external forces are conservative. Then we have potentials $V_i^{\text{ext}}(\mathbf{x}_i)$ for the external forces, i.e.,

$$\mathbf{F}_i^{\text{ext}} = -\nabla_i V_i^{\text{ext}},$$

where ∇_i has the same meaning as in §4.4. For each pair (i,j) of particles there is also a potential $V_{ij}(\mathbf{x}_i, \mathbf{x}_j)$ for the internal force, so that

$$\mathbf{F}_{ij} = -\nabla_i V_{ij} \qquad \text{and} \qquad \mathbf{F}_{ji} = -\nabla_j V_{ij}.$$

$\mathscr{N}$ ***III*** is automatically satisfied if V_{ij} is a function of $|\mathbf{x}_i - \mathbf{x}_j|$, as is usually the case. Let

$$V(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \sum_{i=1}^{n} V_i^{\text{ext}}(\mathbf{x}_i) + \sum\sum V_{ij}(\mathbf{x}_i, \mathbf{x}_j)$$

where the double sum is over *unordered pairs* (i,j) of particles (i.e., we count each pair of particles only once in the sum). Then, noting that for instance $\nabla_k V_i^{\text{ext}}(\mathbf{x}_i) = \mathbf{0}$ if $k \neq i$,

$$\begin{aligned} -\nabla_k V &= -\nabla_k V_k^{\text{ext}} - \sum_{j\neq k} \nabla_k V_{kj}(\mathbf{x}_k, \mathbf{x}_j) \\ &= \mathbf{F}_k^{\text{ext}} + \sum_{j\neq k} \mathbf{F}_{kj} \\ &= m_k \ddot{\mathbf{x}}_k. \end{aligned}$$

Hence

$$\begin{aligned} & & \sum_k m_k \dot{\mathbf{x}}_k \cdot \ddot{\mathbf{x}}_k &= -\sum_k \dot{\mathbf{x}}_k \cdot \nabla_k V \\ &\Longrightarrow & \frac{\mathrm{d}T}{\mathrm{d}t} &= -\frac{\mathrm{d}V}{\mathrm{d}t} & &\text{(by the chain rule for } V\text{)} \\ &\Longrightarrow & T + V &= \text{const.} \end{aligned}$$

9.4 The Two-Body Problem

Consider two particles moving under a mutual force which depends only on their relative position, i.e.,

$$m_1\ddot{\mathbf{x}}_1 = \mathbf{F}_{12} = -\mathbf{F}_{21} = -m_2\ddot{\mathbf{x}}_2$$

where $\mathbf{F}_{12}$ is a function of $\mathbf{x}_1 - \mathbf{x}_2$. The centre of mass is at

$$\mathbf{X} = \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2}$$

and we know from §9.1 that $\ddot{\mathbf{X}} = \mathbf{0}$. Let

$$\mathbf{r} = \mathbf{x}_1 - \mathbf{x}_2$$

be the relative position vector; then

$$\begin{aligned} \mathbf{x}_1' &= \mathbf{x}_1 - \mathbf{X} \\ &= \mathbf{x}_1 - \frac{m_1\mathbf{x}_1 + m_2\mathbf{x}_2}{m_1 + m_2} \end{aligned}$$

$$
\begin{aligned}
&= \frac{m_2}{m_1+m_2}(\mathbf{x}_1 - \mathbf{x}_2) \\
&= \frac{m_2}{m_1+m_2}\mathbf{r},
\end{aligned}
$$

and $\mathbf{x}_2' = -\big(m_1/(m_1+m_2)\big)\mathbf{r}$ similarly. Hence

$$
\begin{aligned}
m_1\ddot{\mathbf{x}}_1 &= m_1(\ddot{\mathbf{X}} + \ddot{\mathbf{x}}_1') \\
&= \frac{m_1 m_2}{m_1+m_2}\ddot{\mathbf{r}}.
\end{aligned}
$$

Defining

$$
\boxed{\mu = \frac{m_1 m_2}{m_1+m_2}}
$$

to be the *reduced mass* we obtain

$$
\boxed{\mu\ddot{\mathbf{r}} = \mathbf{F}_{12}(\mathbf{r});}
$$

therefore the relative position vector behaves as if it were a particle of mass μ moving in the same force field.

Example: two bodies orbit each other under mutual gravitational attraction. The relative position vector behaves like a particle of mass μ:

$$
\begin{aligned}
\mu\ddot{\mathbf{r}} &= -\frac{Gm_1m_2}{|\mathbf{r}|^2}\hat{\mathbf{e}}_r \\
\implies \qquad \ddot{\mathbf{r}} &= -\frac{Gm_1m_2}{\mu r^2}\hat{\mathbf{e}}_r = -\frac{G(m_1+m_2)}{r^2}\hat{\mathbf{e}}_r.
\end{aligned}
$$

Comparing this with a standard planetary orbit from Chapter 8 (where the equation of motion was $\ddot{\mathbf{r}} = -(GM/r^2)\hat{\mathbf{e}}_r$, M being the mass of the Sun), we see that the relative position vector describes an ellipse with period

$$
T = \frac{2\pi}{\sqrt{G(m_1+m_2)}} R^{3/2}
$$

where R is the mean separation (by replacing "GM" in §8.5 by "$G(m_1+m_2)$").

The two-body problem can therefore be solved completely. The three-body problem, with three mutually gravitating bodies, was studied extensively in the 19$^{\text{th}}$ and early 20$^{\text{th}}$ centuries, in particular by Poincaré, without success. It was eventually *proved* that the system cannot be solved analytically (the proof rests on the fact that there are not enough constants of the motion): in fact it is typically chaotic.
