

Chapter 6

Central Forces

6.1 Central Forces in Two Dimensions

A *central force* is one which is always directed towards or away from a fixed point, which we may take as our origin. In two dimensions, it is convenient to use plane polar coordinates (r, θ) : then a central force \mathbf{F} must be of the form $\mathbf{F} = f(r, \theta)\hat{\mathbf{e}}_r$. Using \mathcal{NII} and the results for acceleration in plane polar coordinates from §2.1, we see that

$$\begin{aligned}f(r, \theta) &= m(\ddot{r} - r\dot{\theta}^2), \\ 0 &= m(2\dot{r}\dot{\theta} + r\ddot{\theta}).\end{aligned}$$

Note that

$$\frac{d}{dt}(r^2\dot{\theta}) = 2r\dot{r}\dot{\theta} + r^2\ddot{\theta} = r(2\dot{r}\dot{\theta} + r\ddot{\theta});$$

so

$$0 = \frac{m}{r} \frac{d}{dt}(r^2\dot{\theta}) \quad \Longrightarrow \quad \boxed{r^2\dot{\theta} = \text{const.} \equiv h.}$$

h is related to the angular momentum: in fact it is the magnitude of the angular momentum vector per unit mass, as we shall see in the next section. It is conserved because there is no torque (i.e., no force perpendicular to the particle's position vector).

Suppose that f is actually a function of r alone, not of θ , and let $V(r)$ be the corresponding potential, i.e., $f(r) = -V'(r)$. Then

$$\begin{aligned}f &= m(\ddot{r} - r\dot{\theta}^2) = m\left(\ddot{r} - \frac{h^2}{r^3}\right) \tag{6.1} \\ \Longrightarrow \quad -\dot{r}V' &= m\left(\dot{r}\ddot{r} - \dot{r}\frac{h^2}{r^3}\right)\end{aligned}$$

$$\implies -V = m \left(\frac{1}{2} \dot{r}^2 + \frac{h^2}{2r^2} \right) + \text{const.},$$

by integrating with respect to time. Rearranging,

$$\frac{1}{2} m \left(\dot{r}^2 + \frac{h^2}{r^2} \right) + V = \text{const.} \equiv E.$$

This is the energy equation: $\frac{1}{2} m \dot{r}^2$ is the “radial part” of the kinetic energy and $\frac{1}{2} m h^2 / r^2 = \frac{1}{2} m r^2 \dot{\theta}^2$ is the “tangential part”.

Note that we have reduced the problem to a system with one degree of freedom (because we can rewrite (6.1) as $m\ddot{r} = F^*(r)$ where $F^*(r) = f(r) + m h^2 / r^3$), so we can apply the methods of Chapter 4.

6.2 Central Forces in Three Dimensions

In 3D, define the *angular momentum* to be

$$\mathbf{h} = \mathbf{r} \times \mathbf{p} = m \mathbf{r} \times \dot{\mathbf{r}}.$$

Note that

$$\dot{\mathbf{h}} = m \dot{\mathbf{r}} \times \dot{\mathbf{r}} + m \mathbf{r} \times \ddot{\mathbf{r}} = \mathbf{r} \times \mathbf{F}.$$

The RHS is known as the *couple* or *torque*,

$$\mathbf{G} = \mathbf{r} \times \mathbf{F}.$$

A central force in 3D is still defined to be one where $\mathbf{F} \parallel \mathbf{r}$. So under a central force, there is no torque and hence $\dot{\mathbf{h}} = \mathbf{0}$, i.e., \mathbf{h} is a fixed vector. But

$$\mathbf{r} \cdot \mathbf{h} = m \mathbf{r} \cdot (\mathbf{r} \times \dot{\mathbf{r}}) = 0;$$

this is the equation of a plane through the origin. The particle is therefore confined to this plane, and the motion is in fact two-dimensional. So we can use polar coordinates in the plane and the results of the previous section apply.

Now $\mathbf{r} = r \hat{\mathbf{e}}_r$ and $\dot{\mathbf{r}} = \dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta$, so

$$\begin{aligned} \mathbf{h} &= m r \hat{\mathbf{e}}_r \times (\dot{r} \hat{\mathbf{e}}_r + r \dot{\theta} \hat{\mathbf{e}}_\theta) \\ &= m r^2 \dot{\theta} \hat{\mathbf{e}}_r \times \hat{\mathbf{e}}_\theta \\ &= m r^2 \dot{\theta} \hat{\mathbf{e}}_z \end{aligned}$$

where $\hat{\mathbf{e}}_z$ is the unit normal to the plane (since $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are perpendicular, of unit length and both lie in the plane).

This agrees with the result in the previous section that $r^2 \dot{\theta}$ is constant: it is equal to $|\mathbf{h}|/m$.

6.3 Stability of Circular Orbits

Consider motion under a central force $\mathbf{F} = f(r)\hat{\mathbf{e}}_r$: from (6.1) we have

$$f(r) = m \left(\ddot{r} - \frac{h^2}{r^3} \right)$$

where $h = r^2\dot{\theta}$ is constant. Suppose that the particle is moving in a circular orbit of radius a : then the angular velocity $\dot{\theta} = h/a^2$ is also constant, and

$$f(a) = -\frac{mh^2}{a^3}. \quad (6.2)$$

Is this orbit stable? Suppose that the particle is given a small disturbance. Let $r = a + \varepsilon$; then

$$\begin{aligned} f(a + \varepsilon) &= m \left(\ddot{\varepsilon} - \frac{h^2}{(a + \varepsilon)^3} \right) \\ \implies f(a) + \varepsilon f'(a) + O(\varepsilon^2) &= m \left(\ddot{\varepsilon} - \frac{h^2}{a^3} \left(1 + \frac{\varepsilon}{a} \right)^{-3} \right) \\ &= m\ddot{\varepsilon} + f(a) \left(1 - \frac{3\varepsilon}{a} + O(\varepsilon^2) \right), \end{aligned}$$

using (6.2) and Taylor expansions on both sides. To first order we obtain

$$\ddot{\varepsilon} = \frac{1}{m} \left(f'(a) + \frac{3f(a)}{a} \right) \varepsilon.$$

So the orbit is stable iff

$$\boxed{f'(a) + \frac{3f(a)}{a} < 0,}$$

in which case r executes SHM about $r = a$.

We could have obtained the same result by writing $m\ddot{r} = F^*(r)$ as in §6.1 and applying the results of §4.2: $r = a$ is a stable equilibrium iff $F^*(a) = 0$ and $F^{*'}(a) < 0$.

A small disturbance would in fact usually also change the value of h , say to $h + \eta$ where η is small. This leads to the equation

$$\ddot{\varepsilon} = \frac{1}{m} \left(f'(a) + \frac{3f(a)}{a} \right) \varepsilon + \frac{2h\eta}{a^2}$$

to first order (ignoring small terms such as $\varepsilon\eta$). This is still SHM, but plus a particular integral which is just the constant $-2mh\eta/(a^2 f'(a) + 3af(a))$. So the stability criterion is unchanged; the particle simply oscillates about a nearby value of r , i.e., its mean radius is slightly different from the original radius.

Example: $f(r) \propto -1/r^{N-1}$ where N is a positive integer. The stability criterion is then

$$\frac{N-1}{a^N} - \frac{3}{a^N} < 0,$$

i.e., $N < 4$ (or $N \leq 3$, since N is an integer).

In N dimensions, gravity would obey an inverse $(N-1)^{\text{th}}$ power law. It is therefore fortunate for us that circular orbits are stable for $N = 3$; if the Universe had higher dimension then we would not exist!