Chapter 11

Phase-Plane Techniques

11.1 Plane Autonomous Systems

A *plane autonomous system* is a pair of simultaneous first-order differential equations,

$$\dot{x} = f(x, y),$$

$$\dot{y} = g(x, y).$$

This system has an *equilibrium point* (or *fixed point* or *critical point* or *singular point*) (x_0, y_0) when $f(x_0, y_0) = g(x_0, y_0) = 0$.

We can illustrate the behaviour of the system by drawing trajectories (i.e., solution curves) in the (x, y)-plane, known in this context as the *phase plane*. The trajectories in such a *phase portrait* are marked with arrows to show the direction of increasing time. Note that trajectories can never cross, because the solution starting from any point in the plane is uniquely determined: so there cannot be two such solution curves starting at any given point. The only exception is at an equilibrium point (because the solution starting at an equilibrium point is just that single point, so it is no contradiction for two curves to meet there).

11.2 Classification of Equilibria

We can examine the stability of an equilibrium point by setting $x = x_0 + \xi$, $y = y_0 + \eta$ and using Taylor Series in 2D for small ξ and η:

$$\dot{\xi} = f(x_0 + \xi, y_0 + \eta)$$

$$= f(x_0, y_0) + \xi \frac{\partial f}{\partial x}(x_0, y_0) + \eta \frac{\partial f}{\partial y}(x_0, y_0)$$

$$= \xi f_x(x_0, y_0) + \eta f_y(x_0, y_0)$$

and similarly

$$\dot{\eta} = \xi g_x(x_0, y_0) + \eta g_y(x_0, y_0),$$

where we have ignored terms of second order and higher. In matrix notation,

$$\boldsymbol{\xi} = \begin{pmatrix} \xi \\ \eta \end{pmatrix}, \qquad \dot{\boldsymbol{\xi}} = \mathsf{J}\,\boldsymbol{\xi} \tag{11.1}$$

where

$$\boxed{\mathsf{J} = \begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\Bigg|_{(x_0, y_0)} .}$$

Let the eigenvalues of this *stability matrix* J be λ_1, λ_2 with corresponding eigenvectors $\mathbf{e}_1$, $\mathbf{e}_2$. The general solution of (11.1) is

$$\boldsymbol{\xi} = A e^{\lambda_1 t} \mathbf{e}_1 + B e^{\lambda_2 t} \mathbf{e}_2 \tag{11.2}$$

where A, B are arbitrary constants. The behaviour of the solution therefore depends on the eigenvalues.

- Two real, positive eigenvalues (say $\lambda_1 > \lambda_2 > 0$).

  The solution for $\boldsymbol{\xi}$ moves outwards in both the $\mathbf{e}_1$ and $\mathbf{e}_2$ directions (though more quickly in the $\mathbf{e}_1$ direction). Therefore $|\boldsymbol{\xi}|$ increases exponentially with time; trajectories move away from the equilibrium point (x_0, y_0). This is an *unstable node.*

- Two real, negative eigenvalues (say $\lambda_1 < \lambda_2 < 0$).

  In this case, $|\boldsymbol{\xi}|$ decreases exponentially and trajectories move towards the equilibrium point. This is a *stable node.* The phase portrait is identical to that of an unstable node with the arrows reversed.

- Two real eigenvalues of opposite sign (say $\lambda_1 < 0$, $\lambda_2 > 0$).

  Trajectories move inwards along $\mathbf{e}_1$ but outwards along $\mathbf{e}_2$. Unless the initial value of $\boldsymbol{\xi}$ lies *exactly* parallel to $\mathbf{e}_1$, the solution will eventually move away from the equilibrium point, so it is unstable. This is a *saddle point.*

- Complex eigenvalues (a conjugate pair, say $\alpha \pm \mathrm{i}\beta$).

  We can rewrite the general solution (11.2) in the form

$$\begin{aligned}\boldsymbol{\xi} &= \mathrm{e}^{\alpha t}\bigl\{A\bigl(\cos(\beta t) + \mathrm{i}\sin(\beta t)\bigr)\mathbf{e}_1 + B\bigl(\cos(\beta t) - \mathrm{i}\sin(\beta t)\bigr)\mathbf{e}_2\bigr\} \\ &= \mathrm{e}^{\alpha t}\bigl(\mathbf{a}\cos(\beta t) + \mathbf{b}\sin(\beta t)\bigr)\end{aligned}$$

  where $\mathbf{a} = A\mathbf{e}_1 + B\mathbf{e}_2$, $\mathbf{b} = \mathrm{i}(A\mathbf{e}_1 - B\mathbf{e}_2)$. (Of course, $\mathbf{a}$ and $\mathbf{b}$ must be real, because $\boldsymbol{\xi}$ is real at all times, even though $\mathbf{e}_1$, $\mathbf{e}_2$, A and B are complex.)

  Therefore the solution spirals around the equilibrium point (starting with a displacement in the direction of $\mathbf{a}$ at $t = 0$, then moving round to $\mathbf{b}$, then $-\mathbf{a}$, then $-\mathbf{b}$, before getting back to $\mathbf{a}$ and starting again). If $\alpha > 0$ then $|\boldsymbol{\xi}|$ increases with each loop, so this is an *unstable focus*; if $\alpha < 0$ then it is a *stable focus*. If $\alpha = 0$ (i.e., if the eigenvalues are purely imaginary) then the solution just goes round and round a closed loop (an ellipse with semi-axes $\mathbf{a}$ and $\mathbf{b}$): this is a *centre*.

  The sense of the rotation (i.e., either clockwise or anticlockwise) is best discovered on a case-by-case basis (usually by considering the sign of either f or g close to the equilibrium point).

Note that $\mathbf{e}_1$ and $\mathbf{e}_2$ are not necessarily orthogonal. Similarly, for complex eigenvalues $\alpha \pm \mathrm{i}\beta$, $\mathbf{a}$ and $\mathbf{b}$ are not necessarily orthogonal; so in this case the diagrams above might need to be skewed. In the case of a centre ($\alpha = 0$) this would result in a sheared ellipse: but a sheared ellipse is still an ellipse (albeit one with different axes of symmetry). The diagrams are qualitatively correct.

Using the Trace and Determinant

In practice, it is not necessary to find the exact values of the eigenvalues and eigenvectors; the *signs* of the eigenvalues (or of their real parts) are all that is required to perform the categorisation. The characteristic equation for J is

$$(f_x - \lambda)(g_y - \lambda) - f_y g_x = 0$$

or, equivalently,

$$\lambda^2 - T\lambda + \Delta = 0$$

where $T = f_x + g_y$ is the trace and $\Delta = f_x g_y - f_y g_x$ the determinant of J. We note that the eigenvalues are real iff $T^2 - 4\Delta \geqslant 0$; and that the product of the eigenvalues is Δ while their sum is T. This enables us to deduce the required signs.

Summary of Results

$\boldsymbol{\Delta}$	$\boldsymbol{T^2 - 4\Delta}$	**Eigenvalues of J**	**Classification**
$-$ve	[+ve]	Real, opposite signs	Saddle
+ve	+ve	Real, same signs	Node: $\begin{cases} T < 0 & \text{stable} \\ T > 0 & \text{unstable} \end{cases}$
+ve	$-$ve	Complex conjugate pair	Focus: $\begin{cases} T < 0 & \text{stable} \\ T = 0 & \text{centre} \\ T > 0 & \text{unstable} \end{cases}$

In the above analysis we have ignored the following degenerate cases:

- One of the eigenvalues is zero, which occurs iff $\Delta = 0$. In this situation we would have to consider second-order terms in the Taylor series, which we could neglect in the analysis above. These terms could either stabilise or destabilise the equilibrium point.
- The two eigenvalues are equal (and real), which occurs iff $T^2 - 4\Delta = 0$. In this situation there may be two non-parallel eigenvectors, just as normal; or there may be only one eigenvector, in which case the general solution is not of the form given above. In either case, the classification still holds: the equilibrium point is an unstable node if $\lambda_1 = \lambda_2 > 0$ and a stable node if $\lambda_1 = \lambda_2 < 0$. The phase diagram looks somewhat different, however: a *star node* (in the case of two non-parallel eigenvectors) or an *inflected node* (in the case of only one).

11.3 The Phase Plane for a Conservative System

A second-order differential equation for a variable $x(t)$ can always be converted to two first-order differential equations by defining $y = \dot{x}$. For example, a general force equation in one dimension,

$$m\ddot{x} = F(x, \dot{x}),$$

can be converted to

$$\dot{x} = y,$$

$$\dot{y} = \frac{1}{m} F(x, y)$$

which is of the form given in §11.1 for a plane autonomous system.

Now consider a conservative force field in one dimension described by a potential $V(x)$ so that $F = -V'(x)$. Then

$$\dot{x} = y,$$
$$\dot{y} = -\frac{1}{m}V'(x).$$

Equilibrium points occur where $V'(x_0) = 0$ and $y_0 = 0$ (so they are all on the x-axis). At such a point,

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -V''(x_0)/m & 0 \end{pmatrix},$$

so $T = 0$ and $\Delta = V''(x_0)/m$. Hence, using the summary table in the previous section, all equilibrium points must be either saddles (if $V''(x_0) < 0$) or centres (if $V''(x_0) > 0$). This analysis agrees with the stability analysis of §4.2, with the following phase plane diagrams locally (i.e., close to the equilibrium point):

We also have the energy equation

$$\tfrac{1}{2}my^2 + V(x) = E.$$

This equation, for different values of the constant E, defines the trajectory curves in the (x, y)-plane. Trajectories for this system are therefore symmetric in y (which enables us to draw the diagrams above, knowing that the directions given by the eigenvectors must also be reflectionally symmetric in the x-axis).

On a given trajectory, at any value of x there are just two values of y, one positive and one negative. If a trajectory is bounded then it must have $y = 0$ at each end, at which points it joins up. Trajectories for a conservative system must therefore be either *closed* (i.e., they come back to where they started) or *unbounded*.

Example: the Duffing oscillator for a particle of unit mass,

$$\ddot{x} = x - x^3.$$

This corresponds to $V(x) = -\frac{1}{2}x^2 + \frac{1}{4}x^4$. We will have a saddle at $x = 0$ and centres at $x = \pm 1$. The phase portrait is as follows:

Example: a simple pendulum of length l (as in §2.5) has

$$\ddot{\theta} = -\frac{g}{l}\sin\theta.$$

Let $y = \dot{\theta}$, so that

$$\dot{\theta} = y,$$

$$\dot{y} = -\frac{g}{l}\sin\theta.$$

The stable equilibria are at $\theta = 2n\pi$ and the unstable ones at $\theta = (2n+1)\pi$ $(n \in \mathbb{Z})$, by considering

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -(g/l)\cos\theta & 0 \end{pmatrix}.$$

The phase diagram has this form:

Around stable equilibria, the pendulum oscillates back and forth (e.g., curve A). This motion is known as *libration*. If the pendulum has sufficiently large energy then it can instead undergo *rotation* (e.g., curve C) where it always has the same sign of $\dot{\theta}$.

The curves which join the saddle points (e.g., B) are known as *separatrices* because they separate the phase plane into regions containing these two different kinds of motion. They correspond physically to the (highly unlikely) motion where the pendulum starts vertically upwards then executes precisely one revolution, ending vertically upwards again.

11.4 Damped Systems

Consider a simple pendulum with damping:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - k\dot{\theta} \tag{11.3}$$

where k is a small positive constant. Letting $y = \dot{\theta}$ we have

$$\begin{aligned}\dot{\theta} &= y,\\ \dot{y} &= -\frac{g}{l}\sin\theta - ky.\end{aligned}$$

The equilibrium points are still where $\sin\theta = 0$; we have

$$\mathsf{J} = \begin{pmatrix} 0 & 1 \\ -(g/l)\cos\theta & -k \end{pmatrix}.$$

At $\theta = 2n\pi$, $T = -k$ and $\Delta = g/l$, corresponding to a stable focus (from the table in §11.2, assuming that k is small enough that $k^2 < 4g/l$). At $\theta = (2n+1)\pi$, $T = -k$ and $\Delta = -g/l$ so we have a saddle.

Because this system is not conservative (the force depends on $\dot{\theta}$ as well as θ, prohibiting the existence of a potential $V(\theta)$), the solution curves are not symmetric in y and are not closed. The phase portrait is as follows:

Note that if we define energy in the same way as for an undamped pendulum,

$$E = \tfrac{1}{2}ml^2\dot{\theta}^2 - mgl\cos\theta,$$

then

$$\begin{aligned}\frac{\mathrm{d}E}{\mathrm{d}t} &= ml^2\dot{\theta}\ddot{\theta} + mgl\dot{\theta}\sin\theta \\ &= -mkl\dot{\theta}^2 && \text{(from (11.3))} \\ &\leqslant 0,\end{aligned}$$

so the energy is decreasing. (Without the damping term we would have had $\dot{E} = 0$.) This allows us to deduce the portrait above from the undamped portrait of §11.3, because the solution continuously moves to curves of lower energy.

A more interesting example is the van der Pol oscillator

$$\ddot{x} = -x - (x^2 - 1)\dot{x}.$$

For large x we have damping (because $x^2 - 1 > 0$); but for small x we have "negative damping" ($x^2 - 1 < 0$). The only equilibrium is at the origin and is an unstable focus. The phase portrait looks like this:

So *all* trajectories (whether starting from small or large x, except for the one at the origin itself) tend towards a *limit cycle*. Therefore, after an initial transient the system always settles down into this finite amplitude oscillation. If we had studied only the linear stability near the origin we would have concluded that disturbances grow exponentially; the fact that this growth is in fact limited is useful in practical situations.

11.5 The Forced Pendulum

Finally, add time-periodic forcing to the pendulum:

$$\ddot{\theta} = -\frac{g}{l}\sin\theta - k\dot{\theta} + F\cos\Omega t$$

where Ω is the forcing frequency and F its amplitude. This is no longer an autonomous system (unlike in §11.1), because t appears in the governing equation in addition to θ and $\dot{\theta}$. The phase "plane" becomes three-dimensional and consequently much more elaborate: for instance, trajectories can now twist around each other, which is prevented in two dimensions by the rule that trajectories cannot cross except at equilibrium points.

When F is small, we obtain a resonant response as described in §2.2. But for larger forcing amplitudes the system exhibits much more complicated behaviour because the nonlinearity of the $\sin\theta$ term interacts with the forcing. Eventually chaos results.

For small values of F, a graph of oscillation amplitude $|A|$ versus forcing frequency Ω has a simple resonant peak. As F is increased, the oscillations of the pendulum become larger; the restoring force $-(g/l)\sin\theta$ is smaller than the linear approximation $-(g/l)\theta$ that applies for small F, so the period of the oscillations increases. This means that the forcing frequency Ω required to produce resonance decreases for larger $|A|$, so the peak bends backwards.

When F becomes larger still, the oscillation amplitude can take more than one value for a range of Ω. Suppose that we slowly increase Ω from small values: then the amplitude will move along the lower curve shown until it has to "jump" to the upper curve. But when Ω is slowly decreased again the amplitude moves back in a *different* way, sticking to the upper curve until it has to "fall" down to the lower one. This phenomenon, whereby the solution of a system depends not just on its parameters but also on the *history* of those parameters, is known as *hysteresis*.

Many nonlinear systems also exhibit *chaos*, as does the forced pendulum for very large F. In a chaotic system, trajectories never settle into a limit cycle or other repeating pattern. Furthermore, the solutions are very sensitive to the initial conditions, so that if we start with two sets of conditions very close to each other (say $\leqslant 10^{-10}$ apart) at $t = 0$, within a relatively short time the solutions are a long way apart (several orders of magnitude greater, e.g., $\geqslant 1$). This makes the solution effectively unpredictable, even on a very accurate computer.