Chapter 2

Dynamical Examples

2.1 Velocity and Acceleration in Plane Polar Coordinates

In two dimensions (x, y) it is sometimes more convenient to perform calculations using plane polar coordinates (r, θ) where $x = r\cos\theta$, $y = r\sin\theta$. We define the standard vectors

$$\hat{\mathbf{e}}_r = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}, \qquad \hat{\mathbf{e}}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}.$$

It is clear that $\mathbf{r} = r\hat{\mathbf{e}}_r$, i.e., $\hat{\mathbf{e}}_r = \mathbf{r}/|\mathbf{r}|$, so $\hat{\mathbf{e}}_r$ is the unit radial vector. Note that $\hat{\mathbf{e}}_\theta$ is also a unit vector, perpendicular to $\hat{\mathbf{e}}_r$ (because $\hat{\mathbf{e}}_\theta \,.\, \hat{\mathbf{e}}_r = 0$): it is therefore the unit tangential vector, in the direction of increasing θ.

Now $\hat{\mathbf{e}}_r$ and $\hat{\mathbf{e}}_\theta$ are independent of r, but

$$\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = \hat{\mathbf{e}}_\theta, \qquad \frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta} = \begin{pmatrix} -\cos\theta \\ -\sin\theta \end{pmatrix} = -\hat{\mathbf{e}}_r.$$

So

$$\begin{aligned}
& & \mathbf{r} &= r\hat{\mathbf{e}}_r \\
&\implies & \dot{\mathbf{r}} &= \dot{r}\hat{\mathbf{e}}_r + r\frac{\partial \hat{\mathbf{e}}_r}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
&\implies & &\boxed{\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{e}}_r + r\dot{\theta}\hat{\mathbf{e}}_\theta} \\
&\implies & \ddot{\mathbf{r}} &= \ddot{r}\hat{\mathbf{e}}_r + 2\dot{r}\dot{\theta}\hat{\mathbf{e}}_\theta + r\ddot{\theta}\hat{\mathbf{e}}_\theta + r\dot{\theta}\frac{\partial \hat{\mathbf{e}}_\theta}{\partial \theta}\frac{\mathrm{d}\theta}{\mathrm{d}t} \\
&\implies & &\boxed{\ddot{\mathbf{r}} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{e}}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\mathbf{e}}_\theta.}
\end{aligned}$$

Here $\dot\theta$ is the rate of rotation about the origin, known as the *angular velocity* (or strictly the *angular speed*), measured in radians per second. The term *angular frequency*, or simply (but misleadingly) *frequency*, is also used.

Conversion to and from Cartesian Coordinates

If we know $\dot r$ and $\dot\theta$, it is easy to compute $\dot x$ and $\dot y$ using the definitions $x = r\cos\theta$, $y = r\sin\theta$:

$$\dot x = \dot r\cos\theta - r\dot\theta\sin\theta, \qquad \dot y = \dot r\sin\theta + r\dot\theta\cos\theta.$$

Conversely, given $\dot x$ and $\dot y$ we can find $\dot r$ and $\dot\theta$:

$$r^2 = x^2 + y^2 \qquad \Longrightarrow \qquad 2r\dot r = 2x\dot x + 2y\dot y \qquad \Longrightarrow \qquad \boxed{\dot r = \frac{x\dot x + y\dot y}{r}}$$

and

$$\begin{aligned} \tan\theta = \frac{y}{x} \qquad &\Longrightarrow \qquad \dot\theta\sec^2\theta = \frac{x\dot y - y\dot x}{x^2} \\ &\Longrightarrow \qquad \dot\theta = \frac{x\dot y - y\dot x}{x^2(1+\tan^2\theta)} = \frac{x\dot y - y\dot x}{x^2(1+y^2/x^2)} \\ &\Longrightarrow \qquad \boxed{\dot\theta = \frac{x\dot y - y\dot x}{r^2}.} \end{aligned}$$

Each of these equations can easily be understood in terms of taking components of the vector velocity. For instance,

$$\dot x = \dot r\cos\theta - r\dot\theta\sin\theta$$

results from resolving the two components of the vector velocity $\dot{\mathbf{r}} = \dot r\hat{\mathbf{e}}_r + r\dot\theta\hat{\mathbf{e}}_\theta$ in the x-direction.

Example: If $\dot x = x + y$ and $\dot y = y - x$ then

$$\dot r = \frac{x(x+y) + y(y-x)}{r} = r,$$

$$\dot\theta = \frac{x(y-x) - y(x+y)}{r^2} = -1.$$

Circular Motion

Consider now motion in a circle, so that r is constant, and let $\omega = \dot\theta$ (not necessarily constant) be the angular velocity. Then

$$\dot{\mathbf{r}} = \omega r\hat{\mathbf{e}}_\theta$$

and

$$\ddot{\mathbf{r}} = -\omega^2 r\hat{\mathbf{e}}_r + \dot{\omega} r\hat{\mathbf{e}}_\theta.$$

The term $\dot{\omega} r\hat{\mathbf{e}}_\theta$ represents acceleration along the curve itself, i.e., it represents the effect of varying angular velocity. However, the term $-\omega^2 r\hat{\mathbf{e}}_r$, which is known as the *centripetal acceleration*, is present even if ω is constant; this represents the acceleration towards the centre of the circle which is required simply to keep the motion constrained to the circle. Let $v = \omega r$ be the speed; then the magnitude of the centripetal acceleration can be written alternatively as $\omega^2 r$, v^2/r or ωv.

Example: A car is travelling on a hump-backed bridge whose surface may be modelled as an arc of circle with radius of curvature R. At the crest of the bridge, a policeman observes that the car becomes temporarily airborne. Should he issue a speeding ticket?

At the crest of the bridge, the forces acting on the car are gravity vertically downwards and a possible normal reaction vertically upwards; however, because the car has left the road, it is clear that the normal reaction must be zero. The acceleration downwards (towards the centre of the circle) is therefore g, which must be insufficient to constrain the car to the circle; i.e., $g < v^2/R$. So $v > \sqrt{gR}$ and the policeman should issue a ticket if $\sqrt{gR}$ is greater than the local speed limit.

2.2 Simple Harmonic Motion

Free Motion

Consider a particle of mass m attached to a spring with spring constant k:

$$m\ddot{x} = -kx$$

so	$x = A\sin(\sqrt{k/m}\,t) + B\cos(\sqrt{k/m}\,t)$	for arbitrary (real) constants A, B
or	$\quad = R\sin(\sqrt{k/m}\,t + \theta_0)$	for suitable R, θ_0
or	$\quad = \alpha \mathrm{e}^{\mathrm{i}\sqrt{k/m}\,t} + \beta \mathrm{e}^{-\mathrm{i}\sqrt{k/m}\,t}$	for suitable (complex) α, β.

The angular frequency of the oscillation is $\omega = \sqrt{k/m}$ and the period is $2\pi/\omega$.

Other examples include the pendulum, a magnetic dipole and an electrical circuit.

Damped Motion

Consider the same system with linear friction, imposed perhaps by a "dashpot" or shock absorber:

$$m\ddot{x} = -kx - c\dot{x}$$

where c is a positive constant. Let $\omega = \sqrt{k/m}$, the *natural frequency*, and let $\gamma = c/(2m)$; then the roots of the auxiliary equation are $-\gamma \pm \sqrt{\gamma^2 - \omega^2}$. There are three cases:

- $\gamma > \omega$ – *overdamped*. Both roots are real and negative, so the solution contains two exponentially decaying terms.

- $\gamma = \omega$ – *critically damped*. The roots are equal and

$$x = (A + Bt)\mathrm{e}^{-\gamma t}.$$

- $\gamma < \omega$ – *underdamped*. We obtain a decaying oscillation

$$x = R\mathrm{e}^{-\gamma t}\cos(\sqrt{\omega^2 - \gamma^2}\, t + \theta_0).$$

The optimal design for a car suspension is critically damped: overdamping gives a hard ride, whereas underdamping causes prolonged oscillations.

Forced Damped Motion

Suppose that we now apply an additional periodic force $F\cos\Omega t$ to the particle, so that

$$m\ddot{x} = -kx - c\dot{x} + F\cos\Omega t.$$

The general solution consists of two parts: (a) the particular integral; and (b) the complementary function, which we found above and which *always* decays as $t \to \infty$. We are interested in the long-term solution, so we disregard (b), which is the *transient*, and only consider (a), the *forced response*. Like the forcing, this will be periodic with angular frequency Ω; if Ω is close to the natural frequency $\omega = \sqrt{k/m}$ then we will observe *resonance*, especially if the damping is light.

2.3 Uniform Electromagnetic Fields

A charged particle in a uniform magnetic field $\mathbf{B}$ moves in a helix. For example, consider an electron with charge $-e$. The Lorentz force law and $\mathscr{N}\boldsymbol{II}$ give that

$$m\ddot{\mathbf{x}} = -e\dot{\mathbf{x}} \times \mathbf{B}.$$

By rotating the axes, we can arrange that $\mathbf{B}$ is parallel to the z-axis; then

$$m\begin{pmatrix}\ddot{x}\\ \ddot{y}\\ \ddot{z}\end{pmatrix} = -e\begin{pmatrix}\dot{x}\\ \dot{y}\\ \dot{z}\end{pmatrix} \times \begin{pmatrix}0\\ 0\\ B\end{pmatrix},$$

i.e.,

$$m\ddot{x} = -eB\dot{y}, \tag{2.1}$$

$$m\ddot{y} = eB\dot{x},$$

$$m\ddot{z} = 0. \tag{2.2}$$

Equation (2.2) gives $z = vt + z_0$ where v and z_0 are constants. Let $X = \dot{x}$, $Y = \dot{y}$ and $\omega = eB/m$; then

$$\dot{X} = -\omega Y, \quad \dot{Y} = \omega X \qquad \Longrightarrow \qquad \ddot{X} = -\omega^2 X$$

which has solution

$$X = R\sin(\omega t + \theta_0)$$

where R and θ_0 are arbitrary constants. Integrating with respect to t,

$$x = r\cos(\omega t + \theta_0) + x_0$$

where $r = -R/\omega$ and x_0 is an arbitrary constant. Equation (2.1) now gives that

$$\dot{y} = -\frac{\ddot{x}}{\omega} = \omega r\cos(\omega t + \theta_0),$$

so

$$y = r\sin(\omega t + \theta_0) + y_0.$$

where y_0 is another constant. Hence the complete solution is

$$\mathbf{x} = \begin{pmatrix}r\cos(\omega t + \theta_0)\\ r\sin(\omega t + \theta_0)\\ vt\end{pmatrix} + \mathbf{x}_0$$

where $\mathbf{x}_0 = (x_0, y_0, z_0)$ is a constant position vector.

Note that $(x-x_0)^2+(y-y_0)^2 = r^2$, so in the x–y plane the motion is a circle of radius r with centre (x_0, y_0) and angular frequency $\omega = eB/m$ (the "cyclotron frequency"). But the electron also moves at constant speed v in the z-direction, resulting in a helix.

If an electric field $\mathbf{E}$ is also present then

$$m\ddot{\mathbf{x}} = -e(\mathbf{E} + \dot{\mathbf{x}} \times \mathbf{B}),$$

i.e.,

$$\begin{aligned} m\ddot{x} &= -eE_1 - eB\dot{y} \\ m\ddot{y} &= -eE_2 + eB\dot{x} \\ m\ddot{z} &= -eE_3. \end{aligned} \tag{2.3}$$

A similar method to that above, with the addition of some particular integrals, now gives

$$\begin{aligned} x &= r\cos(\omega t + \theta_0) + \frac{E_2}{B}\,t + x_0, \\ y &= r\sin(\omega t + \theta_0) - \frac{E_1}{B}\,t + y_0 \end{aligned}$$

and (2.3) gives

$$z = z_0 + vt - \frac{eE_3}{2m}\,t^2.$$

So as well as the helical motion there is an acceleration $-eE_3/m$ in the $\mathbf{B}$-direction, i.e., an acceleration

$$-\frac{e\mathbf{E}\,.\,\mathbf{B}}{m|\mathbf{B}|^2}\,\mathbf{B},$$

combined with a horizontal drift of constant velocity

$$\begin{pmatrix} E_2/B \\ -E_1/B \\ 0 \end{pmatrix} = \frac{\mathbf{E} \times \mathbf{B}}{|\mathbf{B}|^2}.$$

The advantage of writing the results in vector notation, rather than in coordinates, is that they are then true in *any* coordinate system (and not just the one we have been using in which $\mathbf{B}$ is parallel to the z-axis).

2.4 Projectiles

Linear Friction using Vectors

A ball is thrown from the origin with initial velocity $\mathbf{V}$ and experiences linear air friction. The equation of motion is

$$m\ddot{\mathbf{x}} = m\mathbf{g} - k\dot{\mathbf{x}}$$

where k is a constant, i.e.,

$$\ddot{\mathbf{x}} + (k/m)\dot{\mathbf{x}} = \mathbf{g}.$$

We can solve this just as we would a scalar differential equation. Set $\mathbf{v} = \dot{\mathbf{x}}$ and use an integrating factor $\mathrm{e}^{kt/m}$:

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{v}\mathrm{e}^{kt/m}) = \mathbf{g}\mathrm{e}^{kt/m} \\ \implies \qquad & \mathbf{v}\mathrm{e}^{kt/m} = \frac{m\mathbf{g}}{k}\mathrm{e}^{kt/m} + \mathbf{c}_1 \end{aligned}$$

where $\mathbf{c}_1 = \mathbf{V} - m\mathbf{g}/k$ from the initial condition. Hence

$$\begin{aligned} & \mathbf{v} = \frac{m\mathbf{g}}{k} + \left(\mathbf{V} - \frac{m\mathbf{g}}{k}\right)\mathrm{e}^{-kt/m} \\ \implies \qquad & \mathbf{x} = \frac{m\mathbf{g}}{k}t - \frac{m}{k}\left(\mathbf{V} - \frac{m\mathbf{g}}{k}\right)\mathrm{e}^{-kt/m} + \mathbf{c}_2, \end{aligned}$$

where $\mathbf{c}_2 = (m/k)(\mathbf{V} - m\mathbf{g}/k)$.

Quadratic Friction

A particle of mass m falling vertically downwards under gravity with
quadratic air resistance obeys

$$m\ddot{z} = -mg + k\dot{z}^2$$

where k is a constant and z is the height of the particle at time t.

Note that the particle has a *terminal velocity*: when its speed is $\sqrt{mg/k}$, i.e., $\dot{z} = -\sqrt{mg/k}$, the forces of gravity and air resistance are in balance and it accelerates no more. In fact it never quite reaches this speed.

2.5 The Simple Pendulum

Using the results of §2.1, we see that the acceleration of a pendulum bob of length l at angle θ to the vertical is $l\ddot{\theta}$ tangentially and $l\dot{\theta}^2$ along the string towards the point of suspension.

Resolving in these directions,

$$ml\ddot{\theta} = -mg\sin\theta, \tag{2.4}$$

$$ml\dot{\theta}^2 = T - mg\cos\theta. \tag{2.5}$$

For small oscillations we linearise (2.4) using $\sin\theta \sim \theta$ for $\theta \ll 1$, so

$$\ddot{\theta} = -\frac{g}{l}\,\theta.$$

This is SHM with frequency $\sqrt{g/l}$ and period $2\pi\sqrt{l/g}$.

To find the period of the nonlinear motion, when θ is not necessarily small, we can use energy methods: see §4.1.

Tension in the String

We can also find the tension T. Multiplying (2.4) by $\dot{\theta}/m$ and integrating,

$$l\dot{\theta}\ddot{\theta} = -g\dot{\theta}\sin\theta \qquad \Longrightarrow \qquad \tfrac{1}{2}l\dot{\theta}^2 = g\cos\theta + c$$

where c is a constant which can be found from the initial conditions. (This idea in fact arises from the energy methods of §4.1.) Then from (2.5),

$$T = ml\dot{\theta}^2 + mg\cos\theta = 3mg\cos\theta + 2mc.$$

2.6 Varying Mass

Avalanches

We consider a massively simplified model of an avalanche which nevertheless gives useful results. Take a layer of snow of density ρ and thickness h on the side of a mountain of slope α. As the avalanche descends it sweeps up all the snow in its path into a single "particle". We assume it is of constant width L (into the page on our diagram).

The mass of the avalanche is

$$m(t) = \rho h L x(t)$$

and from $\mathscr{N}\boldsymbol{II}$ (assuming no friction),

$$\frac{\mathrm{d}}{\mathrm{d}t}(mv) = mg\sin\alpha.$$

Using $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\mathrm{d}x}{\mathrm{d}t}\frac{\mathrm{d}}{\mathrm{d}x}$,

$$\begin{aligned}
& & v\frac{\mathrm{d}}{\mathrm{d}x}(\rho h L x v) &= \rho h L x g \sin\alpha \\
&\implies & xv\frac{\mathrm{d}}{\mathrm{d}x}(xv) &= x^2 g\sin\alpha \\
&\implies & \tfrac{1}{2}(xv)^2 &= \tfrac{1}{3}x^3 g\sin\alpha + \text{const.} \\
&\implies & v &= \sqrt{\tfrac{2}{3}xg\sin\alpha} \\
&\implies & \frac{\mathrm{d}x}{\sqrt{x}} &= \sqrt{\tfrac{2}{3}g\sin\alpha}\,\mathrm{d}t \\
&\implies & x &= \tfrac{1}{6}g\sin\alpha\, t^2.
\end{aligned}$$

This is equivalent to uniform acceleration of $\frac{1}{3}g\sin\alpha$, i.e., gravity is reduced by a factor of 3.

Note that this method of using $\mathscr{N}\boldsymbol{II}$ in the form $\mathbf{F} = \dot{\mathbf{p}}$ only works because *all* other particles besides the one we are interested in have zero momentum. Normally we need to use a different approach, taking every particle into account.

Rockets

A rocket has mass $m(t)$, velocity $\mathbf{v}(t)$ and is acted on by a force $\mathbf{F}(t)$. It expels mass at a velocity $-\mathbf{u}$ (usually constant) relative to itself.

In a time interval δt, the velocity changes to $\mathbf{v} + \delta\mathbf{v}$ and the mass to $m + \delta m$. A small amount $-\delta m$ is expelled (note that $\delta m < 0$) at velocity $\mathbf{v} - \mathbf{u} + O(|\delta\mathbf{v}|)$ (i.e., somewhere between $\mathbf{v} - \mathbf{u}$ and $\mathbf{v} + \delta\mathbf{v} - \mathbf{u}$). $\mathscr{N}\boldsymbol{II}$ tells us that

$$\mathbf{p}(t + \delta t) - \mathbf{p}(t) = \mathbf{F}\,\delta t$$

as $\delta t \to 0$, so

$$\big\{(m+\delta m)(\mathbf{v}+\delta\mathbf{v}) + (-\delta m)(\mathbf{v}-\mathbf{u}+O(|\delta\mathbf{v}|))\big\} - m\mathbf{v} = \mathbf{F}\,\delta t$$

$$\implies \qquad m\mathbf{v} + \mathbf{v}\,\delta m + m\,\delta\mathbf{v} - \mathbf{v}\,\delta m + \mathbf{u}\,\delta m - m\mathbf{v} = \mathbf{F}\,\delta t$$

since we may ignore second-order terms. Dividing by δt and taking the limit,

$$\boxed{m\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \mathbf{u}\frac{\mathrm{d}m}{\mathrm{d}t} = \mathbf{F}.}$$

This is the *rocket equation.*

Example: if the rocket has initial mass m_0 and burns mass at a constant rate α, then $\dot{m} = -\alpha$ and $m = m_0 - \alpha t$. If it is launched vertically from rest then

$$(m_0 - \alpha t)\dot{v} - \alpha u = -mg = -(m_0 - \alpha t)g$$

$$\implies \qquad \dot{v} = -g + \frac{\alpha u}{m_0 - \alpha t}$$

$$\implies \qquad v = -gt - u\ln(m_0 - \alpha t) + c$$

where $c = u \ln m_0$ from the initial conditions. Hence

$$v = -gt - u\ln\left(1 - \frac{\alpha t}{m_0}\right)$$

$$\implies \qquad z = -\tfrac{1}{2}gt^2 + \frac{um_0}{\alpha}\left\{\left(1-\frac{\alpha t}{m_0}\right)\ln\left(1-\frac{\alpha t}{m_0}\right) + \frac{\alpha t}{m_0}\right\}.$$

Note that we must require $\dot{v} > 0$ at $t = 0$ in order for the rocket to lift off: hence $-g + \alpha u/m_0 > 0$, i.e., $\alpha > m_0 g/u$. If this is true then in fact $\dot{v} > 0$ for all $t < m_0/\alpha$ (by which time the fuel must have run out and our equation of motion has stopped being valid).