Chapter 3

Dimensional Analysis

3.1 Power Laws

It is not possible to add together a length and an area meaningfully. Similarly, if x is a length then e^x is physically meaningless, because

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots$$

and we would be adding length to area to volume, etc. (What could $e^{2\,\mathrm{cm}}$ mean anyway?) So we can only add together quantities with the same dimensions: for instance in Pythagoras's theorem $a^2 = b^2 + c^2$ all quantities have the dimensions of area, i.e., length^2. In any equation, the dimensions of the LHS must match those of the RHS, to ensure that it is true in *any* system of units.

In fact, quantities with dimensions occur only in *power laws*, for instance volume = length^3. Any other function of a dimensional quantity (such as e^{x^2}, $\ln x$ or $\operatorname{sech}^{-1} x$ where x has dimensions) is meaningless. Thus $\sin\alpha$ only makes sense if α is dimensionless, e.g., if $\alpha = xy/(\pi r^2)$ where x, y and r are lengths.

This simple fact makes it possible to obtain some surprisingly significant results about how physical quantities depend on each other.

We can prove, formally, the result that dimensional quantities occur only in power laws. Let x be a length and y be a physical variable related to x via the law $y = f(x)$ for some function f. For any two lengths x_1 and x_2 we take it as axiomatic that the ratio

$$\frac{y_1}{y_2} = \frac{f(x_1)}{f(x_2)}$$

is dimensionless and therefore independent of our system of units. That is, if we were to rescale the values of x_1 and x_2 by some factor α, say, then the ratio would be unchanged; hence

$$\frac{f(x_1)}{f(x_2)} = \frac{f(\alpha x_1)}{f(\alpha x_2)}$$

for any α. (If we changed our units of length from metres to centimetres, for instance, we would have $\alpha = 100$.)

Differentiating this expression with respect to α we obtain

$$x_1 f(\alpha x_2) f'(\alpha x_1) = x_2 f(\alpha x_1) f'(\alpha x_2)$$

for any α, x_1 and x_2. Setting $\alpha = 1$, $x_1 = x$ and $x_2 = 1$ gives

$$\frac{x f'(x)}{f(x)} = \frac{f'(1)}{f(1)} \equiv k;$$

that is to say, $xf'(x)/f(x)$ is a constant. Integrating this with respect to x gives $\ln f(x) = k \ln x + c$, i.e., $f(x) \propto x^k$. This vindicates our statement.

The argument can easily be extended to more general situations involving other dimensions as well as length.

We can associate with any physical quantity, y say, a set of dimensions denoted by $[y]$ in terms of mass M, length L, time T and possibly others based on the SI system (such as temperature Θ). Charge Q is often included in this list of basic dimensions (even though, according to the SI system, we should strictly use current I instead).

For example, velocity has dimensions LT^{-1}; force has dimensions MLT^{-2} (because $\mathbf{F} = m\mathbf{a} = m\,\mathrm{d}^2\mathbf{x}/\mathrm{d}t^2$); and density has dimensions ML^{-3}.

Once we have done this, we can write down a power law relationship between y and the dimensional parameters a, b, c, ... on which it depends:

$$y = Ca^\alpha b^\beta c^\gamma \ldots$$

where C is a dimensionless constant and α, β, γ, ... are unknown constant exponents. By considering the powers of each dimension M, L, T, etc., on both sides, we can find equations connecting α, β and γ etc. There are three possibilities:

- No solutions for α, β, γ, ... We must have forgotten an important parameter.
- A unique solution. This is lucky and tells us everything about the relationship between y and a, b, c, ... except for one unknown constant C.
- Many solutions. We can still gain useful information in the form of a functional relationship: see §3.2 below.

Example: A pendulum of length l with a bob of mass m executes small oscillations. What is the angular frequency ω?

We know that ω must depend on l, m and g. So posit

$$\omega = Cl^\alpha m^\beta g^\gamma.$$

The dimensions are

$$[l] = L, \qquad [m] = M, \qquad [g] = LT^{-2} \qquad \text{and} \qquad [\omega] = T^{-1}.$$

So

$$T^{-1} = (L)^{\alpha}(M)^{\beta}(LT^{-2})^{\gamma}$$

and hence

$$\left.\begin{aligned} \alpha + \gamma &= 0 \\ \beta &= 0 \\ -2\gamma &= -1 \end{aligned}\right\} \qquad \Longrightarrow \qquad \alpha = -\tfrac{1}{2}, \quad \beta = 0, \quad \gamma = \tfrac{1}{2}.$$

The solution for α, β, γ is unique in this case, so $\omega = C\sqrt{g/l}$. In fact we know from §2.5 that $C = 1$.

3.2 Dimensionless Parameters

When there are many possible solutions for the putative exponents in a supposed power law of the form considered in §3.1, the relationship between the physical quantities may in fact have a more complicated form. We cannot determine the exact nature of the relationship, but we can, nonetheless, show that its functional form must depend only on one or more *dimensionless parameters*, as in the following example.

Example: A ball of mass m is thrown vertically upwards with speed V and experiences quadratic air resistance with coefficient k. How high does it go?

Let h be the height and try

$$h = Cm^{\alpha}g^{\beta}k^{\gamma}V^{\delta}.$$

We know the dimensions (in particular, $[k] = [\text{force}]/[\text{velocity}]^2 = (MLT^{-2})/(LT^{-1})^2 = ML^{-1}$), so we obtain

$$\begin{aligned} \alpha + \gamma &= 0, \\ \beta - \gamma + \delta &= 1, \\ -2\beta - \delta &= 0. \end{aligned}$$

These do *not* have a unique solution; but one solution is $\alpha = 1$, $\beta = 0$, $\gamma = -1$, $\delta = 0$, corresponding to m/k. So consider $h\big/(m/k)$; this must be dimensionless. But dimensionless quantities can only be formed from other dimensionless quantities, so now start again and look for *dimensionless* combinations of m, g, k and V. These would have

$$\begin{aligned} \alpha + \gamma &= 0, \\ \beta - \gamma + \delta &= 0, \\ -2\beta - \delta &= 0 \end{aligned}$$

from which we deduce that $\alpha = \beta = -\gamma$ and $\delta = -2\alpha$, i.e., our dimensionless combination is $(mg/kV^2)^\alpha$. Hence $\lambda \equiv mg/kV^2$ is essentially the only independent dimensionless parameter and we *must* have

$$\frac{h}{m/k} = f\left(\frac{mg}{kV^2}\right), \qquad \text{i.e.,} \qquad h = \frac{m}{k} f\left(\frac{mg}{kV^2}\right) \tag{3.1}$$

where f is some unknown function.

Note that f need not be just a simple power law, because its argument is dimensionless. In fact, solving the ball's equation of motion explicitly shows that f is the function

$$f(\lambda) = \tfrac{1}{2}\ln(1 + \lambda^{-1}),$$

but we cannot find this by dimensional analysis.

In this example we said that *one* combination of parameters with the same dimensions as h is m/k, and considered $h/(m/k)$ as a dimensionless quantity. But we could have chosen a different combination, for instance V^2/g, with the same dimensions as h and considered $h/(V^2/g)$ instead. Would this have led to a different answer?

We would have obtained

$$h = \frac{V^2}{g}\hat{f}\left(\frac{mg}{kV^2}\right)$$

where $\hat{f}$ is some (different) unknown function. But, given any function $\hat{f}(\lambda)$, define

$$\tilde{f}(\lambda) = \frac{\hat{f}(\lambda)}{\lambda};$$

then we see that

$$h = \frac{V^2}{g}\frac{mg}{kV^2}\tilde{f}\left(\frac{mg}{kV^2}\right) = \frac{m}{k}\tilde{f}\left(\frac{mg}{kV^2}\right)$$

where $\tilde{f}$ is some unknown function, which is the same answer as before.

Instead of writing

$$\frac{h}{m/k} = f\left(\frac{mg}{kV^2}\right)$$

in (3.1) we could equally have written

$$g\left(\frac{mg}{kV^2}, \frac{kh}{m}\right) = 0$$

where g is an unknown function. This is achieved simply by defining, for example, $g(x,y) = y - f(x)$. There are occasions on which this alternative form is more convenient.

There are many other suitable definitions of g apart from the example $y - f(x)$ given above: for instance $y^2 - [f(x)]^2$ or $\ln\big[(1+y)/(1+f(x))\big]$.

The alternative form is in fact more general. Given $y = f(x)$ it is always possible to obtain $g(x,y) = 0$ by making the definition for g above. However, given an arbitrary equation $g(x,y) = 0$ it is not always possible to reverse the process and write $y = f(x)$ (for instance, in the case $g(x,y) = (x-1)^2 - (y-1)^2$, where there are in general two solutions for y for any given x, or the case $g(x,y) = x^3$ where y does not even feature!).

Suppose now that instead of throwing the ball vertically upwards, we had thrown it at an angle θ to the horizontal. Since angles are dimensionless, we would have *two* independent dimensionless quantities, mg/kV^2 and θ; hence

$$h = \frac{m}{k} F\left(\frac{mg}{kV^2}, \theta\right)$$

where F is an unknown function of two variables. In fact, F is *not* known analytically!

3.3 Non-Dimensionalisation

Consider the example of §3.2 again, where the ball is thrown at an angle θ. If the ball has position vector $\mathbf{x} = (x, y)$, then its equation of motion is

$$m\ddot{\mathbf{x}} = m\mathbf{g} - k|\dot{\mathbf{x}}|\dot{\mathbf{x}},$$

i.e.,

$$m\ddot{x} = -k\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x},$$

$$m\ddot{y} = -mg - k\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y};$$

at $t = 0$ we have $x = y = 0$, $\dot{x} = V\cos\theta$, $\dot{y} = V\sin\theta$.

We note that a combination of the parameters with dimensions of length is m/k; and of time, $m/(kV)$. So *scale* x, y and t by defining

$$X = \frac{x}{m/k}, \qquad Y = \frac{y}{m/k}, \qquad T = \frac{t}{m/(kV)}.$$

Then X, Y and T are dimensionless variables. Derivatives can be calculated using the chain rule:

$$\dot{x} = \frac{\mathrm{d}T}{\mathrm{d}t}\frac{\mathrm{d}x}{\mathrm{d}T} = \frac{kV}{m}\frac{\mathrm{d}}{\mathrm{d}T}\left(\frac{m}{k}X\right) = VX'$$

where a prime denotes $\mathrm{d}/\mathrm{d}T$. Repeating this procedure we obtain

$$\ddot{x} = \frac{kV^2}{m}X''.$$

So

$$kV^2X'' = -k\sqrt{V^2X'^2 + V^2Y'^2}\,VX',$$

$$kV^2Y'' = -mg - k\sqrt{V^2X'^2 + V^2Y'^2}\,VY'.$$

Defining $\lambda = mg/kV^2$ we obtain

$$X'' = -\sqrt{X'^2 + Y'^2}\,X',$$

$$Y'' = -\lambda - \sqrt{X'^2 + Y'^2}\,Y',$$

with $X = Y = 0$ and $X' = \cos\theta$, $Y' = \sin\theta$ at $T = 0$. These simultaneous differential equations and their initial conditions have thus been entirely non-dimensionalised, and the solution clearly depends only on the parameters λ and θ. We can solve them numerically (on a computer) for $X(T)$ and $Y(T)$, and find the point T_0 at which $Y'(T_0) = 0$; then the height reached is $h = (m/k)Y(T_0)$.

The value of $Y(T_0)$, which depends only on λ and θ, is in fact the function $F(\lambda, \theta)$ mentioned in §3.2. This method allows us to find the complete set of solutions by exploring only 2 parameters λ and θ instead of a 5D parameter space (m, k, g, V, θ).