

Waves - 12 lectures of 24

Part III

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May 29, 2009

This part of course deals with propagation and scattering of acoustic and electromagnetic waves by inhomogeneous, possibly random, media and by rough surfaces. The direct problem of calculating the scattered field, given an incident field and a scatterer (which could be a surface or an extended medium) will be considered first. The last two chapters are concerned with the inverse problem. The linear approximation to the wave equation for acoustic waves will be used throughout.

This is just a first draft of the material covered in this course. I should very much appreciate being told of any corrections or possible improvements. Comments, please, to O.Rath-Spivack@damtp.cam.ac.uk.

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1 Governing equations for acoustic and electromagnetic waves

The first two sections of this chapter are intended as a reminder of concepts that are probably all, or mostly, already familiar to you, with the purpose of establishing terminology and notation and serving as a quick reference.

1.1 Acoustic Waves

We shall start from the *linearised wave equation* for acoustic waves propagating in a fluid with density ρ and velocity $\mathbf{v}$. For an ideal fluid, i.e. with zero viscosity, and neglecting gravity and any other external forces, conservation of mass and conservation of momentum, together with an appropriate state equation, are expressed to first order by the ***linear acoustic equations***

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla \cdot \mathbf{v}' = 0 \quad (1.1)$$

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} = -\nabla p' \quad (1.2)$$

$$p' = \left(\frac{\partial p}{\partial \rho} \right)_0 \rho' = c^2 \rho'. \quad (1.3)$$

and lead to the **wave equation** for the acoustic pressure:

$$\nabla^2 p - \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} = 0, \quad (1.4)$$

where c denotes the speed of sound in the fluid (medium), and ρc , which is equal to the ratio between pressure and velocity, is called ***characteristic impedance*** of the medium.

The wave equation can be formulated alternatively in terms of a ***velocity potential***. If we take the curl of (1.2) and use the vector identity $\nabla \times (\nabla \varphi) = 0$, valid $\forall \varphi$, it follows that (again dropping all primes)

$$\frac{\partial(\nabla \times \mathbf{v})}{\partial t} = 0,$$

i.e. the ***vorticity*** ($\nabla \times \mathbf{v}$) is constant in time. Therefore the velocity field is irrotational ($\nabla \times \mathbf{v} = 0$) if it is irrotational initially, and we can introduce a velocity potential ϕ by writing

$$\mathbf{v} = \nabla \phi. \quad (1.5)$$

1.1 Acoustic Waves

Note that $\mathbf{v} = \nabla\phi + \mathbf{v}_0$ will apply if the fluid is initially moving with velocity $\mathbf{v}_0$. Substituting (1.5) in (1.2), we obtain

$$p = \rho_0 \frac{\partial\phi}{\partial t}. \quad (1.6)$$

Now, using (1.5), (1.6) and (1.3) in (1.2) gives

$$\nabla^2\phi - \frac{1}{c^2} \frac{\partial^2\phi}{\partial t^2} = 0, \quad (1.7)$$

which is the wave equation in terms of the velocity potential.

A general solution of (1.4) is

$$p = f\left(t - \frac{\xi}{c}\right) + g\left(t + \frac{\xi}{c}\right), \quad (1.8)$$

where f and g are arbitrary functions which will be determined by initial and boundary conditions, and ξ is the coordinate along which the acoustic pressure varies, i.e. the direction along which the acoustic disturbance travels. This solution is the sum of two waves travelling at speed c in the $+\xi$ and $-\xi$ direction respectively.

In an arbitrarily oriented coordinate frame, if $\mathbf{n}$ is the unit vector in the direction of increasing ξ , then at a point $\mathbf{x}$ we can write $\xi = \mathbf{n} \cdot \mathbf{x}$. If one assumes, as is usually appropriate from physical considerations, that there exists a time t_0 in the past before which the wave hasn't arrived and all field quantities are zero (**causality**), then the solution reduces to waves travelling in the positive direction:

$$p = f\left(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c}\right). \quad (1.9)$$

For an acoustic disturbance of constant frequency, the field variables oscillate sinusoidally with time, so

$$p = |A| \cos(\omega t - \varphi) = \text{Re}\{Ae^{(i\varphi - i\omega t)}\}, \quad (1.10)$$

where ω = angular frequency,

φ = phase,

and we have $T = \frac{2\pi}{\omega}$ = period,

$f = \frac{\omega}{2\pi}$ = frequency.

1.1 Acoustic Waves

If a sinusoidal wave $p = |A| \cos(\omega t)$ travels in the $\mathbf{n}$ direction, then we must have $p = f(t - \mathbf{n} \cdot \mathbf{x}/c)$, and consequently

$$p = |A| \cos \left[\omega \left(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c} \right) \right] = \text{Re} \{ e^{-i\omega(t - \frac{\mathbf{n} \cdot \mathbf{x}}{c})} \} = \text{Re} \{ e^{i\mathbf{k} \cdot \mathbf{x} - \omega t} \} , \quad (1.11)$$

where we have used the *wavevector* $\mathbf{k} = \frac{\omega}{c}\mathbf{n}$. The above rightmost expression is the one usually and most conveniently used in practical calculations.

NOTE: Even though the physical quantity is given by the real part only, full complex waves are normally used in calculations, and the real part is subsequently taken as appropriate. Consequently, if the acoustic field is expressed in terms of a complex velocity potential ψ , we should be careful to take

$$\begin{aligned} p &= \text{Re} [i\omega\rho\psi \exp(-i\omega t)] \\ \mathbf{v} &= \text{Re} [\nabla\psi \exp(-i\omega t)] \end{aligned} \quad (1.12)$$

when dealing with real physical quantities.

Any acoustic disturbance $p(\mathbf{x}, t)$ can be written as a superposition of time-harmonic waves 1.10. This can be done using a Fourier transform (as long as $|p(\mathbf{x}, t)|$ and $|p(\mathbf{x}, t)| \in L^2$):

$$p(\mathbf{x}, t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\mathbf{x}, \omega) \exp(-i\omega t) d\omega \quad (1.13)$$

where

$$p(\mathbf{x}, \omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} p(\mathbf{x}, t) \exp(i\omega t) dt \quad (1.14)$$

If we substitute a harmonic wave $p = e^{i\mathbf{k} \cdot \mathbf{x} - \omega t}$ in the wave equation (1.4) (noting that $\text{Re}\{\cdot\}$ and $\frac{\partial}{\partial t}\{\cdot\}$ commute), we obtain

$$\frac{\omega^2}{c^2} p + \nabla^2 p = 0 ,$$

or, by using the *wavenumber* $k = \frac{\omega}{c}$

$$\nabla^2 p + k^2 p = 0 . \quad (1.15)$$

This form of the wave equation, suitable for time-harmonic waves, is usually called the **Helmholtz equation**, or **reduced wave equation**.

When considering time-harmonic problems then, it is usual (and obviously very convenient) to drop the time-dependent part of the wave altogether.

1.1 Acoustic Waves

This is possible, at least for part of the calculations, in the case of a non-monochromatic wave, by decomposing it into monochromatic waves using Fourier analysis. Since the wave equation is linear, each Fourier component obeys the Helmholtz equation, and the total field can be reconstructed after solving the scattering problem for whatever boundary conditions on any finite surfaces are appropriate. In this case, though, it is not possible to express the causality condition in the same way as before. Causality then is expressed by the integrability condition implicit in assuming that a Fourier representation of the wave exists. What was introduced as a condition in time (initial value), and cannot in that form be readily applied to a superposition of stationary waves, is equivalent to a condition in space (boundary condition at infinity):

$$p(\mathbf{x}) = \mathcal{O}(|\mathbf{x}|^{-1/2}) \quad (1.16)$$

or, more usually:

$$|\mathbf{x}| \left(\frac{\partial p(\mathbf{x})}{\partial |\mathbf{x}|} - ikp(\mathbf{x}) \right) \rightarrow 0 \quad (1.17)$$

uniformly as $|\mathbf{x}| \rightarrow \infty$. This is the ***Sommerfeld radiation condition***, and it expresses the requirement that the field should contain no incoming waves as $|\mathbf{x}| \rightarrow \infty$. In general, integrability, hence causality, will also result in restrictions imposed on the contour chosen for the integration in the complex plane.

1.2 Electromagnetic waves

1.2 Electromagnetic waves

In this section the wave equation obeyed by *electromagnetic waves* is derived, and we introduce the general scattering problem for electromagnetic waves. We shall begin with Maxwell's equations for an electromagnetic field in a generic medium with permittivity ϵ and permeability μ , in SI units (also sometimes called MKS):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \quad (1.18)$$

$$\nabla \cdot \mathbf{B} = 0 \quad (1.19)$$

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} + \mathbf{J} \quad (1.20)$$

$$\nabla \cdot \mathbf{D} = \rho, \quad (1.21)$$

Here $\mathbf{E}$ is the electric field intensity, $\mathbf{B}$ is the magnetic induction, $\mathbf{H}$ is the magnetic field intensity, $\mathbf{D}$ is the so-called electric displacement, $\mathbf{J}$ is the current density, and ρ is the electric charge density. These quantities are related by

$$\mathbf{D} = \epsilon \mathbf{E} + \mathbf{P} \quad (1.22)$$

$$\mathbf{B} = \mu \mathbf{H} + \mathbf{M}, \quad (1.23)$$

where $\mathbf{P}$ is the electric polarization and $\mathbf{M}$ the magnetization.

In free space, we have $\mathbf{P} = \mathbf{0}$ and $\mathbf{M} = \mathbf{0}$, and Maxwell's equations reduce to

$$\nabla \times \mathbf{E} = -\mu_0 \frac{\partial \mathbf{H}}{\partial t} \quad (1.24)$$

$$\nabla \cdot \mathbf{H} = 0 \quad (1.25)$$

$$\nabla \times \mathbf{H} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \quad (1.26)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad (1.27)$$

where ϵ_0 and μ_0 are the permittivity and permeability of free space respectively.

It is straightforward to see from the Maxwell equations that there exist ***scalar and vector potentials*** for the electromagnetic field. Since $\nabla \cdot \mathbf{B} = 0$, $\exists$ a vector field $\mathbf{A}$ such that

$$\mathbf{B} = \nabla \times \mathbf{A}. \quad (1.28)$$

Using this in the first of Maxwell's equations shows that $\mathbf{E}$ must satisfy:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \quad (1.29)$$

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where V is a scalar field. $\mathbf{A}$ and V are not unique. It is always possible to find an arbitrary scalar Φ such that the vector

$$\mathbf{A}_0 = \mathbf{A} - \nabla\Phi$$

also satisfies (1.28) giving the same $\mathbf{B}$, and the scalar

$$V_0 = V + \frac{\partial\Phi}{\partial t}$$

gives the same $\mathbf{E}$. This is a *gauge transformation*, and any particular choice of Φ is a *choice of gauge*.

We shall see that the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ obey a wave equation equivalent to that derived in section 1.1 for acoustic waves. A similar equation can also be derived for the scalar and vector potentials Φ and $\mathbf{A}$.

Let us derive the wave equation first in free space, i.e. and in the case when there are no charges nor currents: $\rho = 0, \mathbf{J} = \mathbf{0}$. We shall start with equation (1.20), which in this case becomes:

$$\nabla \times \mathbf{B} = \mu_0\epsilon_0 \frac{\partial\mathbf{E}}{\partial t} \quad (1.30)$$

Noting that $\nabla \times \{\cdot\}$ and $\frac{\partial}{\partial t}\{\cdot\}$ commute, if we now apply $\frac{\partial}{\partial t}$ to (1.30), and use equation (1.18), we obtain

$$\nabla \times (\nabla \times \mathbf{E}) = \mu_0\epsilon_0 \frac{\partial^2\mathbf{E}}{\partial t^2} \quad (1.31)$$

and, since $\nabla \cdot \mathbf{E} = 0$ in this case, and $\mu_0\epsilon_0 = c^{-2}$, where c is the speed of light, we arrive at the wave equation for $\mathbf{E}$

$$\nabla^2\mathbf{E} - \frac{1}{c^2} \frac{\partial^2\mathbf{E}}{\partial t^2} = 0. \quad (1.32)$$

It is straightforward to derive a wave equation of the same form for the magnetic field $\mathbf{B}$. A wave equation for $\mathbf{E}$ can be similarly derived in the more general case where charges and currents are present, and, for the case of a homogeneous and isotropic medium, has the form:

$$\nabla^2\mathbf{E} - \frac{1}{c^2} \frac{\partial^2\mathbf{E}}{\partial t^2} = \epsilon_0^{-1}\nabla\rho + \mu_0 \frac{\partial\mathbf{J}}{\partial t}, \quad (1.33)$$

where the r.h.s. represents source terms due to charges and currents. A similar equations for $\mathbf{B}$ also applies.

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In an inhomogeneous medium the permittivity and permeability are space-dependent, and if the medium is also anisotropic, then it has different properties in different directions and its permittivity and permeability are tensors, $\bar{\epsilon}(x, y, z)$ and $\bar{\mu}(x, y, z)$. In this case the wave equation will be a more complicated expression, since we will not be able to factorise ϵ and μ in the same way.

We notice here that the vector product $\mathbf{E} \times \mathbf{H}$ has the dimensions of an energy flux. It is indeed taken as the energy flow at a point (even though it is not unique), and is called **Poynting vector**:

$$\mathbf{S} = \mathbf{E} \times \mathbf{H} = \frac{1}{\mu} \mathbf{E} \times \mathbf{B} \quad (1.34)$$

The Poynting vector gives the direction of the energy flow. For a general time-harmonic field $\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t + \theta_E}\}$ and $\mathbf{H}(\mathbf{r}, t) = \text{Re}\{\mathbf{H}(\mathbf{r})e^{-i\omega t + \theta_H}\}$ we can see that the time-averaged energy flux:

$$\bar{S} = \frac{1}{T} \int (\mathbf{E} \times \mathbf{H}) dt$$

is given by half the real part of the complex Poynting vector:

$$\bar{S} = \frac{1}{2} \text{Re}[\mathbf{E}(\mathbf{r}) \times \mathbf{H}^*(\mathbf{r})] \quad (1.35)$$

Similarly to acoustic plane waves, an electromagnetic plane wave shall be written $\mathbf{E}(\mathbf{r}, t) = \mathbf{E}_0(t)e^{i\mathbf{k} \cdot \mathbf{r}}$, from which we can see (from Maxwell equations) that for plane waves the energy flow is perpendicular to the wavefront, and the energy travels in the direction of the wavevector $\mathbf{k}$. Note that, even though the functional form of an electromagnetic plane wave is the same as that of an acoustic plane wave, electromagnetic waves are *vector* waves, so all the equations are vector equations.

For a time-harmonic field $\mathbf{E}(\mathbf{r}, t) = \text{Re}\{\mathbf{E}(\mathbf{r})e^{-i\omega t}\}$ we can derive, as in the case of acoustic waves, a reduced wave equation: the Helmholtz equation for electromagnetic waves

$$\nabla^2 \mathbf{E}(\mathbf{r}) + k^2 \mathbf{E}(\mathbf{r}) = 0, \quad (1.36)$$

where $k^2 = \omega^2 \mu \epsilon$. The equivalent equation is also satisfied by $\mathbf{H}(\mathbf{r})$.

The **radiation condition** for electromagnetic waves can be expressed (as before) in terms of the scalar and vector potentials, but is usually more conveniently expressed in terms of the field components:

$$|r\mathbf{E}| < K, \quad |r\mathbf{H}| < K$$

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$$r(\mathbf{E} + Z_0 \hat{\mathbf{r}} \times \mathbf{H}) \rightarrow 0, \text{ as } |\mathbf{r}| \rightarrow \infty, \quad (1.37)$$

$$r(\mathbf{H} - \hat{\mathbf{r}} \times \mathbf{E}/Z_0) \rightarrow 0, \text{ as } |\mathbf{r}| \rightarrow \infty, \quad (1.38)$$

where $Z_0 = \sqrt{\mu/\epsilon} = \textit{impedance}$ of the medium.

Maxwell's equations are linear, so if a distribution of charges and currents gives the e.m. field $(\mathbf{E}_1, \mathbf{H}_1)$ and another gives the field $(\mathbf{E}_2, \mathbf{H}_2)$, then the superposition of the two distributions gives the field $(\mathbf{E}_1 + \mathbf{E}_2, \mathbf{H}_1 + \mathbf{H}_2)$, and they all satisfy the wave equation. This property holds exactly for electromagnetic waves, unlike the case of acoustic waves where it is only approximate, when the linearised wave equation is assumed.

Polarized waves

Plane waves solutions of (1.33) or (1.32) and their equivalents for the magnetic field are again fundamental in practical applications, as in the case of acoustic waves, either because only far-field solutions are of interest, or because any wave can be represented as a superposition of plane waves.

Of particular interest are plane waves which are linearly polarized. Two kinds of linear polarizations are possible. Let's take Cartesian coordinates and a plane wave with direction of propagation $\mathbf{k}$ in the (x, y) -plane. Then, either the electric vector $\mathbf{E}$ is parallel to the z -coordinate:

$$\mathbf{E} = \hat{\mathbf{z}} E_z, \quad \text{E-polarization} \quad (1.39)$$

or TM wave

or:

$$\mathbf{H} = \hat{\mathbf{z}} H_z, \quad \text{H-polarization} \quad (1.40)$$

or TE wave .

When talking of “*direction of polarization*”, one normally refers to the direction of $\mathbf{E}$ (but note that the opposite convention is sometime found in the literature). It is immediately apparent that in many scattering problems with linearly polarized waves, the vector wave equation will reduce to a scalar equation for either E_z or H_z .

For example, if a TM wave is incident on a surface that can be described by $S = f(\rho, \phi)$ in cylindrical polar coordinates, independently of z , then for this scattering problem the incident field is given by

$$\mathbf{E}^{inc} = \hat{\mathbf{z}} E_z^{inc}, \quad \mathbf{H}^{inc} = -\frac{i}{kZ} \left(\frac{\partial E_z^{inc}}{\partial y} \hat{\mathbf{x}} - \frac{\partial E_z^{inc}}{\partial x} \hat{\mathbf{y}} \right), \quad (1.41)$$

where $Z = \sqrt{\mu/\epsilon}$ is the surface impedance, and depends on the properties of the two media and the surface, and usually varies with the incoming field

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at each point. In general, Z is also a function of frequency and angle of incidence.

Since the boundary conditions are independent of z , then the scattered field must also be E -polarized, and of the form

$$\mathbf{E}^{sc} = \hat{\mathbf{z}}E_z^{sc}, \quad \mathbf{H}^{sc} = -\frac{i}{kZ} \left(\frac{\partial E_z^{sc}}{\partial y} \hat{\mathbf{x}} - \frac{\partial E_z^{sc}}{\partial x} \hat{\mathbf{y}} \right), \quad (1.42)$$

therefore the scattering problem reduces to finding the scalar function E_z^{sc} , and is analogous to the problem of an acoustic field scattered by a soft surface. Similarly, the case of H -polarization is analogous to that of an acoustic field scattered by a hard surface. All problems where the scatterer is axisymmetric and the incident electromagnetic field is polarized in the direction parallel to the axis of symmetry therefore reduce to a scalar problem.

1.3 Boundary conditions

1.3 Boundary conditions

The solutions of the various forms of the wave equations will typically have to be found subject to **boundary conditions** on any surface in the problem. A generic surface is an interface between two media, and the boundary conditions must reflect the continuity of actual physical quantities at the interface. The constraints imposed on the solutions of the wave equation at an interface between two fluids will reflect the different characteristic properties of the two fluids or, if the surface delimits a solid object, the properties of the solid object defined by the surface.

In the case of acoustic waves, the b.c. at an interface between two media, let say medium 1 and medium 2 with densities ρ_1 and ρ_2 , must reflect continuity of pressure, which means that there cannot be a net force at the interface, and continuity of the normal component of the velocity, which means that the two media are in contact at the interface (no gaps). These are normally expressed in terms of the velocity potential by the 'jump conditions':

$$\rho_1 \psi_1 = \rho_2 \psi_2 \quad (1.43)$$

$$\frac{\partial \psi_1}{\partial n} = \frac{\partial \psi_2}{\partial n}$$

where the subscripts 1 and 2 refer to the two media, and we take $\mathbf{n}$ as the normal directed into medium 1.

In the case of electromagnetic waves, we can see from Maxwell's equations that there will be discontinuities in some components of the fields, which are related to the surface distribution of charges and currents. We have :

$$\mathbf{E}_2 - \mathbf{E}_1 = (\rho_s/\epsilon)\mathbf{n} \quad (1.44)$$

$$\mathbf{H}_2 - \mathbf{H}_1 = \mathbf{J}_s \times \mathbf{n} , \quad (1.45)$$

(where the subscript s refers to the quantities at the surface) which imply discontinuity in the normal component of the electric field $\mathbf{E}_\perp$ and the tangential component of the magnetic field $\mathbf{H}_\parallel$.

We can derive the appropriate continuity equation for the components of the electromagnetic field by applying the integral theorems of vector calculus to an infinitesimal layer across the interface.

1.3 Boundary conditions

We shall first apply Gauss theorem to $\int \nabla \cdot \mathbf{B} dv$ over an infinitesimal cylinder across the interface, with height δh and basis area δA .

We have

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{B} dv = \iint_{\mathcal{S}} \mathbf{B} \cdot \mathbf{n} ds . \quad (1.46)$$

We then let $\delta h \rightarrow 0$ and assume that $\mathbf{B}$ is constant over the area, since δA is very small, and we can write:

$$\mathbf{B}_1 \cdot \mathbf{n}_1 \delta A + \mathbf{B}_2 \cdot \mathbf{n}_2 \delta A = 0 , \quad (1.47)$$

where $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normals directed into medium 1 and medium 2 respectively, from which we have:

$$(\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{n} = 0 , \quad (1.48)$$

hence $\mathbf{B}_\perp$ (or $\mathbf{H}_\perp$) is continuous.

Similarly, from

$$\iiint_{\mathcal{V}} \nabla \cdot \mathbf{D} dv = \iint_{\mathcal{S}} \mathbf{D} \cdot \mathbf{n} ds = \iiint_{\mathcal{V}} \rho dv \quad (1.49)$$

and defining a **surface charge density** ρ_s by

$$\lim_{\delta h \rightarrow 0} \iiint_{\mathcal{V}} \rho dv = \iint_{\mathcal{S}} \rho_s ds , \quad (1.50)$$

we get

$$(\mathbf{D}_2 - \mathbf{D}_1) \cdot \mathbf{n} = \rho_s . \quad (1.51)$$

To look at the tangential components, let us take an infinitesimal area $\mathcal{S}$ across the interface, with height δh perpendicular to the interface, and sides

1.3 Boundary conditions

δs_1 and δs_2 parallel to the interface. We further denote by $\mathcal{C}$ the total boundary of this area, and by $\mathbf{b}$ its normal.

Using Stokes theorem and Maxwell's equation we have:

$$\iint_S \nabla \times \mathbf{E} \cdot \mathbf{b} ds = \oint_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = - \iint_S \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{b} ds . \quad (1.52)$$

Since the lengths of the parallel sides are very small, we can take them to be equal and take $\mathbf{E}$ to be constant over each side and write:

$$\mathbf{E}_1 \cdot \mathbf{t}_1 \delta s + \mathbf{E}_2 \cdot \mathbf{t}_2 \delta s + \text{“contrib. from } \perp \text{ sides”} = - \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{b} \delta s \delta h \quad (1.53)$$

Now, by letting $\delta h \rightarrow 0$ and noting that

$$\mathbf{t}_1 = -\mathbf{t}_2 = \mathbf{b} \times \mathbf{n} ,$$

we get

$$\mathbf{n} \times (\mathbf{E}_2 - \mathbf{E}_1) = 0 . \quad (1.54)$$

Similarly, and defining a **surface current density** J_s by

$$\lim_{\delta h \rightarrow 0} \iiint_{\mathcal{V}} J dv = \iint_S J_s ds , \quad (1.55)$$

we get

$$\mathbf{n} \times (\mathbf{H}_2 - \mathbf{H}_1) = J_s . \quad (1.56)$$

If the surface at the interface is **perfectly reflecting** (for acoustic waves), then no energy is allowed through, therefore the magnitude of the reflected wave must be equal to the magnitude of the incident wave. This can be achieved with a reflection coefficient which is either +1 or -1, corresponding to the two cases where the total field *at the surface* is twice the incident field or vanishes (and conversely for the normal derivative).

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The equivalent of a perfectly reflecting surface for electromagnetic waves is a **perfectly conducting** surface, where no energy is transmitted because the electric and magnetic fields inside the conductor are zero, and the tangential component of the total electric field at the surface is zero:

$$\mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\mathbf{n} = 0, \quad (1.57)$$

For perfectly reflective or perfectly conductive surfaces the two possible cases are:

Neumann condition, when the normal derivative of the potential field is given at the boundary, i.e., if $\mathbf{n}$ is the unit normal pointing outward from the surface:

$$\frac{\partial \psi(\mathbf{r})}{\partial n} = 0, \quad \mathbf{r} \text{ on } S. \quad (1.58)$$

For acoustic waves, this corresponds to an acoustically hard surface, or in the case of electromagnetic waves in 2D, to a vertically polarized electromagnetic wave on the perfectly conducting surface.

Dirichlet condition, when the value of the potential field is given at the boundary:

$$\psi(\mathbf{r}) = 0, \quad \mathbf{r} \text{ on } S. \quad (1.59)$$

which, for acoustic waves, corresponds to a pressure-release or acoustically soft surface, and in the case of electromagnetic waves corresponds to a horizontally polarized electromagnetic wave in 2D on the perfectly conducting surface.

In most real cases the surface is neither perfectly reflecting, nor perfectly conducting. Both the potential and its normal derivative are different from zero at the boundary, and it is convenient to express the boundary condition as an approximate equation relating these two quantities. This is called **Cauchy condition** (or **Robin**, or **impedance boundary condition**).

In the case of acoustic waves the impedance boundary condition is usually expressed by

$$\frac{\partial \psi}{\partial n}(\mathbf{r}) = iZ(\mathbf{r}, \omega, \theta, \dots)\psi(\mathbf{r}) \quad \mathbf{r} \text{ on } S. \quad (1.60)$$

For electromagnetic waves the boundary condition relates the tangential component of the electric field at the surface to the normal component of the magnetic field at the surface:

$$\mathbf{E} - (\mathbf{E} \cdot \mathbf{n})\mathbf{n} = iZ(\mathbf{r}, \omega, \theta, \dots)\mathbf{n} \times \mathbf{H}, \quad (1.61)$$

The impedance of the surface, Z , depends on the properties of the two media and usually varies with the incoming field at each point. In general, Z is also a function of frequency and angle of incidence.

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The impedance boundary condition can also be expressed as

$$\mathbf{n} \times \nabla \times \mathbf{E}_{\parallel} = iZ\mathbf{n} \times (\mathbf{E} \times \mathbf{n})$$

in a form similar to the one for scalar waves.

1.4 Green's functions

In most problems of practical interest in acoustics, there will be one or more sources of sound, and the space where the problem needs to be solved will include one or more surfaces. Consequently, the differential equation to be solved will be an *inhomogeneous* version of (1.4) or (1.15), and the solutions will be subject to other *boundary conditions* in addition to (1.17). In general the problem in question will then be defined by a differential equation

$$\nabla^2 p(\mathbf{x}, t) + k^2 p(\mathbf{x}, t) = f(\mathbf{x}, t) , \quad (1.62)$$

together with boundary conditions on one or more surfaces and the Sommerfeld conditions. It is usually not easy to find solutions for such boundary value problems, but the task is greatly facilitated by the use of an auxiliary function associated with the differential equation, known as **Green's function**.

In order to illustrate the concept of a Green's function, and provide the means of constructing Green's functions for different problems, let's first write (1.62) in operator form as

$$Lp(\xi) = f(\xi) , \quad (1.63)$$

where L is a linear operator, p the unknown function, and f is a known function determined by the source. The variable ξ denotes a point in an n -dimensional space which can include time as one of the coordinates. The solution of (1.63) can be sought in principle by finding the inverse of the operator L ,

$$p(\xi) = L^{-1}f(\xi) , \quad (1.64)$$

but this is so far not particularly useful in practice. Since L is a differential operator, if L^{-1} exists, it can be reasonably assumed to be an integral operator. If we assume that L^{-1} is an integral operator with kernel K , i.e. such that

$$L^{-1}f(\xi) = \int K(\xi, \eta)f(\eta)d\eta$$

for any functions f defined in the same domain as p , then we can write

$$p(\xi) = LL^{-1}p(\xi) = L \int K(\xi, \eta)p(\eta)d\eta ,$$

Since L is a differential operator with respect to the variable ξ , we can formally write

$$p(\xi) = \int LK(\xi, \eta)p(\eta)d\eta .$$

1.4 Green's functions

This can be true only if

$$LK(\xi, \eta) = \delta(\eta - \xi) , \quad (1.65)$$

in which case we can write the solution to (1.63) as

$$p(\xi) = \int K(\xi, \eta) f(\eta) d\eta \quad (1.66)$$

The kernel K of the operator L^{-1} is called the *Green's function* for the problem and will thereafter be denoted by $G(\xi, \eta)$. We can see from (1.66) that its knowledge allows us to find the solution of the wave equation for any known source $f(\xi)$, at least in principle. Equation (1.65) shows that the Green's function is the field generated by a delta-function inhomogeneity, i.e. the solution of the inhomogeneous wave equation (1.63) with the source term $f = \delta(\eta - \xi)$.

Due to the symmetric property of G :

$$G(\xi, \eta) = G^*(\eta, \xi)$$

This reciprocity relation means that $G(\mathbf{x}, \mathbf{y}, t, t')$ can equivalently represent the field at a point $\mathbf{x}$ due to a 'disturbance' at $\mathbf{y}$, or the field at $\mathbf{y}$ due to a 'disturbance' at $\mathbf{x}$. In other words, the Green's function is unchanged if source and receiver are interchanged. We note that, with regard to the time coordinate, the reciprocity implies time reversal: $G(\mathbf{x}, \mathbf{y}, t, 0) = G(\mathbf{y}, \mathbf{x}, 0, -t)$, so causality is satisfied.

The Green's function defined above is not unique: it is always possible to add to it a solution of the homogeneous wave equation, and the result will of course still satisfy (1.65). The particular solution for the Green's function which is independent of any boundary conditions is called the *free space Green's function*, and shall usually be denoted by $G_0(\xi, \eta)$. Any other Green's function can be written as

$$G(\xi, \eta) = G_0(\xi, \eta) + G_H(\xi, \eta) , \quad (1.67)$$

where $G_H(\xi, \eta)$ is a solution of

$$L(\xi)G(\xi, \eta) = 0 . \quad (1.68)$$

When $G_H(\xi, \eta)$ is chosen to satisfy the boundary conditions for the problem, then $G(\xi, \eta)$ is the exact Green's function for the problem.

We shall derive here the free space Green's function for time-dependent wave equation in 1D, i.e. the function G satisfying:

$$\frac{\partial^2 G(x, t)}{\partial t^2} - c^2 \frac{\partial^2 G(x, t)}{\partial x^2} = \delta(x - y)\delta(t - \tau) \quad (1.69)$$

1.4 Green's functions

If we Fourier transform (1.69) in both space and time, it becomes

$$-\omega^2 \hat{G}(k, \omega) + c^2 k^2 \hat{G}(k, \omega) = e^{iky} e^{-i\omega\tau} , \quad (1.70)$$

so the transform of the required Green's function is given by

$$\hat{G}(k, \omega) = \frac{1}{c^2} \frac{e^{iky} e^{-i\omega\tau}}{k^2 - \omega^2/c^2} , \quad (1.71)$$

and $G(x, t)$ can be obtained by transforming back:

$$G(x, t) = \frac{1}{4\pi^2 c^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{e^{-ik(x-y)} e^{i\omega(t-\tau)}}{k^2 - \omega^2/c^2} dk d\omega \quad (1.72)$$

The integral in (1.72) must be calculated taking care that the contour of integration is chosen in a way that satisfies the causality condition. As discussed in section 1.1, this means requiring that the time-Fourier transformed function $G(x, \omega)$ must be analytic in $\text{Im}(\omega) \leq 0$. Therefore, when integrating in the complex k -plane, we need to take the limit from below at the pole $k = \omega/c$, and the limit from above at the pole $k = -\omega/c$. In the first case the contour will have a small indentation above the pole, in the second case, a small indentation below. With these constraints then, if we first carry out the inverse in k -space we obtain:

$$G(x, \omega) = \frac{1}{4\pi^2 c^2} \int_{-\infty}^{\infty} \frac{e^{-ik(x-y)}}{k^2 - \omega^2/c^2} dk = \frac{e^{-i\omega \frac{|x-y|}{c}}}{4\pi i \omega c} . \quad (1.73)$$

The inverse transform in time then gives:

$$G(x, t) = \frac{1}{4\pi i c} \int_{-\infty}^{\infty} \frac{e^{i\omega(t-\tau-\frac{|x-y|}{c})}}{\omega} d\omega = \frac{1}{2c} H\left(t - \tau - \frac{|x-y|}{c}\right) . \quad (1.74)$$

The time $(t - \tau - \frac{|x-y|}{c})$ is called **retarded time**, and is the time at which the disturbance observed at (x, t) has been emitted by the source at (y) .

In 3 dimensions, the free space Green's function for the time-dependent wave equation is

$$G(x, t) = \frac{1}{4\pi c^2 r} \delta(t - \tau - r/c) , \quad (1.75)$$

where $r = |\mathbf{x} - \mathbf{y}|$,

and the free space Green's function for the Helmholtz equation is

$$G(x, t) = \frac{e^{ikr}}{4\pi r} . \quad (1.76)$$

1.4 Green's functions

The above represents a spherically symmetric wave, and can be derived as the wave generated by a source consisting of an oscillating sphere, in the limiting case where the radius tends to zero. Such source is called a ***point source***, or ***monopole***. In the case of electromagnetic waves, a point source is equivalent to a charge.

It is instructive to consider a source $Q(\mathbf{r})$, uniformly distributed within a sphere. The Helmholtz equation for the wave field is then

$$\nabla^2 p(\mathbf{x}, \omega) + k^2 p(\mathbf{x}, \omega) = Q(\mathbf{x}) . \quad (1.77)$$

This can now be written, using (1.66), as:

$$p(\mathbf{x}, \omega) = \frac{1}{4\pi} \int \frac{e^{ikr}}{r} Q(\mathbf{y}) d\mathbf{y} \quad (1.78)$$

If the radius of the sphere $\mathbf{r}'$ is very small, so $r' \ll r$, then we can expand $(e^{ik|\mathbf{r}-\mathbf{r}'|})/(|\mathbf{r}-\mathbf{r}'|)$ in a power series:

$$\begin{aligned} (e^{ik|\mathbf{r}-\mathbf{r}'|})/(|\mathbf{r}-\mathbf{r}'|) &= \\ \frac{e^{ikr}}{r} - \mathbf{r}' \cdot \nabla \left(\frac{e^{ikr}}{r} \right) &+ \frac{1}{2} (\mathbf{r}' \cdot \nabla)^2 \left(\frac{e^{ikr}}{r} \right) + \dots \end{aligned}$$

If we substitute this expansion in (1.78), we obtain:

$$p = Q_0 \frac{e^{ikr}}{r} + Q_i \frac{e^{ikr}}{r^2} + Q_{ij} \frac{e^{ikr}}{r^3} + \dots \quad (1.79)$$

The coefficients Q_0 , Q_i and Q_{ij} (obtained by integrating over the volume of the sphere containing the sources), are called respectively ***monopole***, ***dipole*** and ***quadrupole*** strength, and the series just obtained ***multipole expansion***.

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

By using the Green's function it is possible to derive an integral form of the Helmholtz equation, which facilitates calculations of sound propagation and scattering, and allows sources and boundary conditions to be treated in a simple and convenient way.

In order to derive this integral equation, we shall first recall the following vector identities. Given any two function f and g , we have:

$$\nabla \cdot (f \nabla g) = f \nabla^2 g + (\nabla f) \cdot (\nabla g) . \quad (V1)$$

If $f \nabla g$ is a vector field continuously differentiable to first order, which we shall denote by $\mathbf{F} = f \nabla g$, then we can apply to it the following theorem, which transforms a volume integral into a surface integral:

Gauss theorem *If V is a subset of $\mathbb{R}^n$, compact and with piecewise smooth boundary S , and $\mathbf{F}$ is a continuously differentiable vector field defined on v , then*

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot \mathbf{n} dS , \quad (V2)$$

where $\mathbf{n}$ is the outward-pointing unit normal to the boundary S .

In $\mathbb{R}^3$, for an $\mathbf{F}_1 = f \nabla g$ and an $\mathbf{F}_2 = g \nabla f$, we have, using V2 and V1:

$$\int_V [f \nabla^2 g + (\nabla f) \cdot (\nabla g)] dV = \int_{\partial V} f \nabla g \cdot \mathbf{n} dS , \quad (1.80)$$

$$\int_V [g \nabla^2 f + (\nabla g) \cdot (\nabla f)] dV = \int_{\partial V} g \nabla f \cdot \mathbf{n} dS , \quad (1.81)$$

and subtracting (1.81) from (1.80) we obtain:

$$\int_V (f \nabla^2 g - g \nabla^2 f) dV = \int_{\partial V} (f \nabla g - g \nabla f) \cdot \mathbf{n} dS . \quad (1.82)$$

This result can be used to solve a general scattering problem, involving one or more sources and write the solution in terms of the (unknown) field and its normal derivative along the boundary. The integral equations obtained can in principle be solved to find these unknown surface field values. This approach applies whether the problem involves an interface with a vacuum or with a second medium.

Consider first a finite region V contained between two smooth closed surfaces S_0 and S_1 , and containing a source $Q(\mathbf{r})$.

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

Let G be the free space Green's function, and ψ the solutions to the inhomogeneous equation

$$\nabla^2 \psi + k^2 \psi = Q(\mathbf{r}) . \quad (1.83)$$

Using the vector identities introduced above, we can write:

$$\int_V (\psi \nabla^2 G - \nabla^2 \psi G) dV = \int_{S_0+S_1} \left(\psi \frac{\partial G}{\partial n} - \frac{\partial \psi}{\partial n} G \right) ds , \quad (1.84)$$

where we have used $d/d\mathbf{n} = \mathbf{n} \cdot \nabla$. If we let the outer surface S_1 go to infinity, then, provided ψ obeys the Sommerfeld boundary condition at infinity, then the integral over S_1 vanishes.

Substituting in (1.84) the expressions for $\nabla^2 \psi$ and $\nabla^2 G$ obtained by the appropriate wave equations, i.e.

$$\begin{aligned} \nabla^2 G &= \delta(\mathbf{r} - \mathbf{r}') - k^2 G \\ \nabla^2 \psi &= Q(\mathbf{r}) - k^2 \psi \end{aligned}$$

we obtain

$$\int_V \psi(\mathbf{r}') \delta(\mathbf{r} - \mathbf{r}') - Q(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' = \int_{S_0+S_1} \left(\psi \frac{\partial G}{\partial n} - \frac{\partial \psi}{\partial n} G \right) ds . \quad (1.85)$$

But

$$\psi_i(\mathbf{r}) = \int_V Q(\mathbf{r}') G(\mathbf{r}, \mathbf{r}') d\mathbf{r}' . \quad (1.86)$$

is the incident field ψ_i inside the volume V . Using this result, then, we can write (1.84) as

$$\psi(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_{S_0} \left[\psi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - \frac{\partial \psi}{\partial n}(\mathbf{r}_0) G(\mathbf{r}, \mathbf{r}_0) \right] d\mathbf{r}_0 . \quad (1.87)$$

This is the **Kirchoff-Helmholtz equation**, an integral (implicit) form of the Helmholtz equation, which is of great practical use in calculating the field induced by sources scattered by finite boundaries.

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

An integral form of the wave equation can be derived also for electromagnetic waves. To this end, we shall use the form of the Gauss theorem as applied to a vector $\mathbf{F} \times \nabla \times \mathbf{G}$:

$$\int_V \nabla \cdot \mathbf{F} \times \nabla \times \mathbf{G} dV = \int_S (\mathbf{F} \times \nabla \times \mathbf{G}) \cdot \mathbf{n} dS, \quad (1.88)$$

Expanding (1.88) gives the identity:

$$\int_V (\nabla \times \mathbf{F} \cdot \nabla \times \mathbf{G} - \mathbf{F} \cdot \nabla \times \nabla \times \mathbf{G}) dV = \int_S (\mathbf{F} \times \nabla \times \mathbf{G}) \cdot \mathbf{n} dS, \quad (1.89)$$

If we then obtain another identity by reversing $\mathbf{F}$ and $\mathbf{G}$ in (1.89), and subtract it from (1.89), we get

$$\int_V (\mathbf{G} \times \nabla \times \nabla \times \mathbf{F} - \mathbf{F} \cdot \nabla \times \nabla \times \mathbf{G}) dV = \int_S (\mathbf{F} \times \nabla \times \mathbf{G} - \mathbf{G} \times \nabla \times \mathbf{F}) \cdot \mathbf{n} dS, \quad (1.90)$$

To derive the integral form of the Helmholtz equation for electromagnetic waves, fictitious magnetic charge densities ρ^* and currents $\mathbf{J}^*$ are introduced, to make the equations symmetric, so Maxwell's equations *for a time-harmonic field* become:

$$\nabla \times \mathbf{E} = i\omega\mu\mathbf{H} - \mathbf{J}^* \quad (1.91)$$

$$\nabla \cdot \mathbf{H} = \frac{\rho^*}{\epsilon} \quad (1.92)$$

$$\nabla \times \mathbf{H} = -i\omega\epsilon\mathbf{E} + \mathbf{J} \quad (1.93)$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon}, \quad (1.94)$$

currents and charges are related by the continuity equations

$$\nabla \cdot \mathbf{J} - i\omega\rho = 0; \quad \nabla \cdot \mathbf{J}^* - i\omega\rho^* = 0 \quad (1.95)$$

and $\mathbf{E}$ and $\mathbf{H}$ satisfy

$$\nabla \times \nabla \times \mathbf{E} - k^2\mathbf{E} = i\omega\mu\mathbf{J} - \nabla \times \mathbf{J}^* \quad (1.96)$$

$$\nabla \times \nabla \times \mathbf{H} - k^2\mathbf{H} = i\omega\epsilon\mathbf{J}^* - \nabla \times \mathbf{J}, \quad (1.97)$$

where $k^2 = \omega^2\epsilon\mu$, as usual. Let us now use (1.90) with $\mathbf{F} = \mathbf{E}$ and $\mathbf{G} = \mathcal{G}\mathbf{a}$, where

$$\begin{aligned} \mathbf{a} &= \text{an arbitrary unit vector} \\ \mathcal{G} &= \frac{e^{ikr}}{r}. \end{aligned}$$

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

Using these definitions, and since $\mathcal{G}$ is a solution of $\nabla^2 \mathcal{G} + k^2 \mathcal{G} = -\delta(r - r')$, we can then write the terms needed in (1.90) as:

$$\begin{aligned}\nabla \times \mathbf{G} &= \nabla \mathcal{G} \times \mathbf{a} \\ \nabla \times \nabla \mathbf{G} &= \mathbf{a} k^2 \mathcal{G} + \nabla(\mathbf{a} \cdot \mathcal{G}) \\ \nabla \times \nabla \mathbf{F} &= k^2 \mathbf{E} + i\omega\mu \mathbf{J} - \nabla \times \mathbf{J}^* .\end{aligned}$$

Using the equations, definitions and identities above in the vector identity (1.88) we obtain:

$$\begin{aligned}& \int_V [i\omega\mu \mathbf{J} \mathcal{G} - \nabla \times \mathbf{J}^* + \frac{1}{\epsilon} \rho \nabla \mathcal{G}] dV = \\ &= \int_S [i\omega\mu(\mathbf{n} \times \mathbf{H}) \mathcal{G} + (\mathbf{n} \times \mathbf{E}) \times \nabla \mathcal{G} + (\mathbf{n} \cdot \mathbf{E}) \nabla \mathcal{G} - (\mathbf{n} \times \mathbf{J}^*) \mathcal{G}] dS ,\end{aligned}\tag{1.98}$$

where the unit vector $\mathbf{a}$ has been dropped, because it is common to all terms and its direction is arbitrary.

By applying the identity

$$\int_V \nabla \times \mathbf{J}^* \mathcal{G} dV = \int_S \mathbf{n} \times \mathbf{J}^* \mathcal{G} dS + \int_V \mathbf{J}^* \times \nabla \mathcal{G} dV ,$$

we can reduce (1.98) to:

$$\begin{aligned}& \int_V [i\omega\mu \mathbf{J} \mathcal{G} - \mathbf{J}^* \times \nabla \mathcal{G} + \frac{1}{\epsilon} \rho \nabla \mathcal{G}] dV = \\ &= \int_S [i\omega\mu(\mathbf{n} \times \mathbf{H}) \mathcal{G} + (\mathbf{n} \times \mathbf{E}) \times \nabla \mathcal{G} + (\mathbf{n} \cdot \mathbf{E}) \nabla \mathcal{G}] dS .\end{aligned}\tag{1.99}$$

Note that the gradient of the Green's function, which appears in the above integral, is given by

$$\nabla \mathcal{G} = \left(\frac{1}{r} - ik \right) \frac{e^{ikr}}{r} \hat{\mathbf{r}} ,$$

which is singular at $r = 0$. Therefore, when calculating the above integrals, we need to exclude a small neighbourhood of $r = 0$, bounded by, e.g., a small sphere of radius r_1 . When we then let the radius tend to zero, the contribution of the surface integral over this small sphere in the r.h.s of (1.99) reduces to $4\pi \mathbf{E}$, because the area over the sphere vanishes with radius as $4\pi r^2$, on the sphere $\hat{\mathbf{r}} = \mathbf{n}$, and $(\mathbf{n} \times \mathbf{E}) \times \mathbf{n} + (\mathbf{n} \cdot \mathbf{E}) \mathbf{n} = \mathbf{E}$.

The field $\mathbf{E}$ at any point $\mathbf{r}'$ in V is then given by

$$\begin{aligned}\mathbf{E}(\mathbf{r}') &= \int_V [i\omega\mu \mathbf{J} \mathcal{G} - \mathbf{J}^* \times \nabla \mathcal{G} + \frac{1}{\epsilon} \rho \nabla \mathcal{G}] dV \\ &- \int_S [i\omega\mu(\mathbf{n} \times \mathbf{H}) \mathcal{G} + (\mathbf{n} \times \mathbf{E}) \times \nabla \mathcal{G} + (\mathbf{n} \cdot \mathbf{E}) \nabla \mathcal{G}] dS ,\end{aligned}\tag{1.100}$$

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

and the equivalent equation for $\mathbf{H}$ is

$$\begin{aligned}\mathbf{H}(\mathbf{r}') &= \int_V [i\omega\epsilon\mathbf{J}^*\mathcal{G} + \mathbf{J} \times \nabla\mathcal{G} + \frac{1}{\mu}\rho^*\nabla\mathcal{G}] dV = \\ &+ \int_S [i\omega\epsilon(\mathbf{n} \times \mathbf{E})\mathcal{G} - (\mathbf{n} \times \mathbf{H}) \times \nabla\mathcal{G} - (\mathbf{n} \cdot \mathbf{H})\nabla\mathcal{G}] dS ,\end{aligned}\quad (1.101)$$

The equations just derived are quite complicated and difficult to calculate. In any practical application it is useful to be able to reduce the calculations to either volume integrals or surface integrals only.

We note that, if all sources can be enclosed within a sphere of finite radius, the field is regular at ∞ and *either side* of S may be chosen as its *interior*, i.e. S may be closed at ∞ . In this case when the surface recedes to ∞ the surface integral vanishes, so when the fictitious magnetic sources (charges and currents) are placed equal to zero we get:

$$\mathbf{E}(\mathbf{r}') = \int_V [i\omega\mu\mathbf{J}\mathcal{G} - \mathbf{J}^* + \frac{1}{\epsilon}\rho\nabla\mathcal{G}] dV \quad (1.102)$$

and

$$\mathbf{H}(\mathbf{r}') = \int_V (\mathbf{J} \times \nabla\mathcal{G}) dV . \quad (1.103)$$

All the above expressions were obtained under the assumption that the medium is homogeneous and isotropic. Some modifications apply for more general media. Useful expressions can be obtained for an inhomogeneous medium where a logarithmic dependence is assumed for the permittivity and permeability gradients:

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{\rho}{\epsilon} - \mathbf{E} \cdot \nabla(\log \epsilon) \\ \nabla \cdot \mathbf{H} &= \frac{\rho^*}{\mu} - \mathbf{H} \cdot \nabla(\log \mu) .\end{aligned}$$

Then the equations equivalent to (1.102) and (1.103) are:

$$\mathbf{E}(\mathbf{r}') = \int_V [(i\omega\epsilon\mathbf{J}^*\mathcal{G} + i\omega\mu(\nabla(\log \mu) \times \mathbf{H})\mathcal{G} - (\mathbf{E} \cdot \nabla(\log \epsilon))\nabla\mathcal{G}] dV \quad (1.104)$$

and

$$\mathbf{H}(\mathbf{r}') = \int_V [\mathbf{J} \times \nabla\mathcal{G} + i\omega\epsilon(\mathbf{E} \cdot \nabla(\log \epsilon))\mathcal{G} - (\mathbf{H} \cdot \nabla(\log \mu))\nabla\mathcal{G}] dV . \quad (1.105)$$

These equations are used in remote sensing to describe variations the $\mathbf{E}$ and $\mathbf{H}$ fields of an electromagnetic wave caused by refraction in a layered inhomogeneous medium, such as the earth atmosphere.

1.5 The Kirchoff-Helmholtz and Stratton-Chu equations

If the volume inside the surface S contains *no charges nor currents* then the volume integrals in (1.100) and (1.101) vanish and the field at any point $\mathbf{r}'$ is given by

$$\mathbf{E}(\mathbf{r}') = \int_S [-i\omega\mu(\mathbf{n} \times \mathbf{H})\mathcal{G} + (\mathbf{n} \times \mathbf{E}) \times \nabla\mathcal{G} + (\mathbf{n} \cdot \mathbf{E})\nabla\mathcal{G}] dS . \quad (1.106)$$

It is natural then to describe this as the the field that would be produced by a distribution of *surface* currents and charges, identified as:

$$\begin{aligned} \mathbf{J}_s &= -\mathbf{n} \times \mathbf{H} \\ \mathbf{J}_s^* &= \mathbf{n} \times \mathbf{E} \\ \rho_s &= -\epsilon\mathbf{n} \cdot \mathbf{E} . \end{aligned}$$

The equivalent expression for the magnetic field in this case is

$$\mathbf{H}(\mathbf{r}') = \int_S [i\omega\epsilon(\mathbf{n} \times \mathbf{E})\mathcal{G} - (\mathbf{n} \times \mathbf{H}) \times \nabla\mathcal{G} - (\mathbf{n} \cdot \mathbf{H})\nabla\mathcal{G}] dS . \quad (1.107)$$

The above equations, (1.106) and (1.107), are usually referred to as the ***Stratton-Chu equations***. These are widely used for problems involving scattering from surfaces and from apertures.

2 Approximations

In general, the solution of most scattering problems can only be expressed analytically as some kind of integral, or as an implicit integral equation. Calculation of the actual values of the field then has to be obtained by computationally intensive numerical solutions. For many problems, though, it is possible to obtain approximate analytical solutions. We shall review the main ones in this chapter.

2.1 Parabolic Equation

Consider first a *scalar plane wave* ψ in *free space* (where we again assume and suppress a time-harmonic variation $e^{-i\omega t}$), with wavenumber k in a two-dimensional medium (x, z) . As before x is horizontal and z is vertical. So ψ obeys the Helmholtz wave equation $(\nabla^2 + k^2)\psi = 0$. Suppose that ψ is propagating at a small angle α to the horizontal, say

$$\psi(x, z) = e^{ik(x \cos \alpha + z \sin \alpha)}. \quad (2.1)$$

Since $\sin \alpha$ is small we can approximate

$$\cos \alpha = \sqrt{1 - \sin^2 \alpha} \cong 1 - \sin^2 \alpha / 2.$$

Now the fastest variation of ψ is close to the x direction, so define the ‘slowly-varying’ part E of ψ by

$$E = \psi e^{-ikx}$$

so that

$$E \cong e^{ik(-x \sin^2 \alpha / 2 + z \sin \alpha)}. \quad (2.2)$$

(E is also referred to as the reduced wave.) It then follows that

$$\frac{\partial E}{\partial x} = \frac{i}{2k} \frac{\partial^2 E}{\partial z^2}. \quad (2.3)$$

This is one form of the ***parabolic wave equation in free space***, and holds for any superposition of plane waves travelling at small angles to the horizontal. (Also referred to as the paraxial or forward scatter equation.)

It is straightforward to write the exact solution of (2.3) in terms of an initial value.

Let E be a field obeying (2.3). Define the Fourier transform of E with respect to z ,

$$\hat{E}(x, \nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} E(x, z) e^{i\nu z} dz. \quad (2.4)$$

2.1 Parabolic Equation

Taking the z -transform of (2.3) gives an equation for $\hat{E}$,

$$\frac{\partial \hat{E}}{\partial x} = -\frac{i\nu^2}{2k} \hat{E}. \quad (2.5)$$

This has solution (in terms of E at vertical plane $x = 0$)

$$\hat{E}(x, \nu) = e^{-i\nu^2 x/2k} \hat{E}(0, \nu). \quad (2.6)$$

Note that equation (2.3) can also be derived by substituting the form $E = \psi e^{ikx}$ into the Helmholtz wave equation for ψ , and neglecting terms of the form $\partial^2 E / \partial x^2$.

We shall now consider the more general case of a *harmonic source in a refractive medium*. Let us consider a point source. It is natural then to use cylindrical coordinates (r, z, θ) , and we shall restrict the problem to one where we assume azimuthal symmetry, so effectively again 2-dimensional, as the field is not dependent on θ . The Helmholtz equation is therefore

$$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial z^2} + k_0^2 n^2 \psi = 0, \quad (2.7)$$

where $k_0 = \omega/c_0$ is a reference wave number, and $n(r, z) = c_0/c(r, z)$ is the index of refraction of the medium.

Let us now rewrite the solution as

$$\psi(r, z) = \frac{u(r, z)}{\sqrt{r}}, \quad (2.8)$$

so we can go on to solve the Helmholtz equation for the wave $u(r, z)$, with the cylindrical spreading removed. In the far field, we obtain

$$\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial z^2} + k_0^2 n^2 u = 0. \quad (2.9)$$

If we now denote the operators appearing in this equation by

$$A = \frac{\partial}{\partial r}, \quad B = \sqrt{\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2}, \quad (2.10)$$

we can factor equation (2.9) as

$$(A - ik_0 B)(A + ik_0 B)u - ik_0[A, B]u = 0. \quad (2.11)$$

For a range-independent medium, where the refractive index does not depend on r , so $n \equiv n(z)$, A and B commute and the last term in (2.10) is zero.

2.1 Parabolic Equation

The remaining term corresponds to factorisation into one outgoing and one incoming wave component. Selecting only the outgoing wave component we obtain the one-way wave equation

$$Au = ik_0 Bu \quad (2.12)$$

or

$$\frac{\partial u}{\partial r} = ik_0 \left(\sqrt{\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2} \right) u \quad (2.13)$$

In order to use this equation in practice, a further approximation is necessary, to resolve the square root operator. If we write B as

$$B = \sqrt{1 + b} , \quad (2.14)$$

where

$$b = \frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2 - 1 , \quad (2.15)$$

then, if b is small, we can Taylor expand B and keep the first 2 terms to give the approximation

$$B \simeq 1 + \frac{b}{2} = 1 + \frac{1}{2k_0^2} \frac{\partial^2}{\partial z^2} + \frac{n^2 - 1}{2} . \quad (2.16)$$

Substituting this expression into (2.13) we obtain a parabolic equation for the 'full' wave in a refractive medium:

$$\frac{\partial u}{\partial r} = \frac{i}{2k_0} \frac{\partial^2 u}{\partial z^2} + \frac{ik_0}{2} (n^2 + 1) u . \quad (2.17)$$

If, as in the free space case, we again separate a 'slowly-varying' part E by defining

$$E = u(r, z) e^{-ikr} = \psi(r, z) \sqrt{r} e^{-ikr} , \quad (2.18)$$

then the Helmholtz equation for E is

$$\frac{\partial^2 E}{\partial r^2} + 2ik_0 \frac{\partial E}{\partial r} - k_0^2 E + \frac{\partial^2 E}{\partial z^2} + k_0^2 n^2 E = 0 , \quad (2.19)$$

and the operator A in the factorisation is

$$A = \frac{\partial}{\partial r} + ik_0 , \quad (2.20)$$

leading to the more usual ***parabolic equation in a refractive medium***:

$$\frac{\partial E}{\partial r} = \frac{i}{2k_0} \frac{\partial^2 E}{\partial z^2} + \frac{ik_0}{2} (n^2 - 1) E . \quad (2.21)$$

2.1 Parabolic Equation

It is seen here that the effect of the medium is contained in the second term on the right hand side. We may loosely think of the first term on the right as the *diffraction term*, and the second as the *scattering term*.

Other forms of the parabolic wave equation can be obtained by using different approximations for the square root operator.

The approximation used above in obtaining the parabolic equation leads to an error proportional to $\sin^4\alpha$.

2.1 Parabolic Equation

Suitable more accurate expansions are obtained in terms of Padé approximants:

Wide-angle methods

Approximating the square root operator with a Padé approximant of the form

$$B = \sqrt{1+b} = \frac{1+pb}{1+qb} \quad (2.22)$$

leads to an error proportional to $\sin^6(\alpha)$.

Approximating the exponential operator which appears in the formal solution directly with a Padé approximant of the form

$$e^{ikx\sqrt{1+b}} \sim 1 + \sum_{l=1}^N \frac{p_l b}{1+q_l b} \quad (2.23)$$

leads to a stable numerical scheme that allows to increase the angular range of validity according to the number N of terms in (2.23).

Summary

- The parabolic wave equation replaces a boundary-value problem with an initial-value problem

- The energy propagates at small angles to a preferred directions (the paraxial direction).

- $\left| \frac{\partial^2 \psi}{\partial x^2} \right| \ll k \left| \frac{\partial \psi}{\partial x} \right|$.

2.2 Born Approximation

- The operators $A = \frac{\partial}{\partial x} + ik_0$ and $B = \sqrt{\frac{1}{k_0^2} \frac{\partial^2}{\partial z^2} + n^2}$ commute (or nearly commute)

equivalently

- The refractive index n is constant (or its variation remains slow on the scale of a wavelength).

The parabolic wave equation can also be derived (rather non-rigorously), as follows.

Again restricting ourselves to 2 dimensions, by regarding the wave as equivalent to the far field of a cylindrically spreading wave in a 3-dimensional medium with cylindrical symmetry, we shall take as starting point the same Helmholtz equation (2.7). As we are in the far field of this wave, we can replace the range r by the horizontal coordinates x , and take z as the vertical coordinate. We then denote by E the slowly varying part of ϕ ,

$$E(x, z) = \psi(x, z) \sqrt{x} e^{-ik_0 x}. \quad (2.24)$$

By substituting (2.24) into (2.7), and neglecting

1. all terms $O\left(x^{-\frac{3}{2}}\right)$ and higher order, since we are in the far field,
2. the term $\frac{\partial^2 E}{\partial x^2}$, which corresponds to slow variation across wavefronts and can be assumed to be small,

we obtain again the parabolic equation

$$\frac{\partial E}{\partial x} = \frac{i}{2k} \frac{\partial^2 E}{\partial z^2} + \frac{ik}{2} (n^2 - 1) E. \quad (2.25)$$

2.2 Born Approximation

The Born approximation is based on expressing the total wave field ψ , which is in general the solution of a scattering problem in a volume with sources and surfaces, as the sum of the incident field plus a 'small' perturbation:

$$\psi = \psi_i + \psi_s, \quad (2.26)$$

The actual solution in this approximation will take various forms, depending on how the perturbation is expressed.

2.2 Born Approximation

We can immediately see how the Born approximation can be applied to the integral form of the wave equation (1.87), to obtain a first Born approximation

$$\psi^{(1)}(\mathbf{r}) = \psi_i(\mathbf{r}) + \int_{S_0} \left[\psi_i(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - \frac{\partial \psi_i}{\partial n}(\mathbf{r}_0) G(\mathbf{r}, \mathbf{r}_0) \right] d\mathbf{r}_0, \quad (2.27)$$

and higher terms can be obtained by iteration.

The Born approximation will only be valid when $\psi_s \ll \psi_i$, which intuitively must apply to some kind of 'weak scattering'. In order to understand better what this means in practice, to relate it to the physical features of a scattering problem, and find boundaries for its range of validity, we shall derive it here for some particular cases.

We shall consider the case where the scattered field is the result of a varying refractive index $n(\mathbf{r})$. The total field satisfies

$$\nabla^2 \psi + k^2(\mathbf{r})\psi = 0. \quad (2.28)$$

We can then write

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})), \quad (2.29)$$

where it is assumed $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (2.28) we get:

$$\nabla^2 \psi + k_0^2(\mathbf{r})\psi = -k_0^2(n^2(\mathbf{r}) - 1)\psi \equiv -V(\mathbf{r})\psi. \quad (2.30)$$

Using (2.26), and the fact that the incident field satisfies

$$\nabla^2 \psi_i + k^2(\mathbf{r})\psi_i = 0, \quad (2.31)$$

we can write the wave equation for the scattered wave

$$\nabla^2 \psi_s + k^2(\mathbf{r})\psi_s = -V(\mathbf{r})\psi. \quad (2.32)$$

We can then solve for ψ_s using the free space Green's function, with $-V(\mathbf{r})\psi$ as the source term

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi(\mathbf{r}')] d\mathbf{r}'. \quad (2.33)$$

But $\psi_s = \psi - \psi_i$, so

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi(\mathbf{r}')] d\mathbf{r}'. \quad (2.34)$$

2.3 Rytov Approximation

We can write the above implicit integral equation as an infinite series of explicit integral equations by forming successive approximations starting from the unperturbed incident field ψ_i :

$$\begin{aligned}\psi^{(0)} &= \psi_i \\ \psi^{(1)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}') \psi^{(0)}(\mathbf{r}')] d\mathbf{r}' \\ \psi^{(2)} &= \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}') \psi^{(1)}(\mathbf{r}')] d\mathbf{r}' \\ \psi^{(3)} &= \dots\end{aligned}$$

The first iteration in this series, $\psi^{(1)}$, is known as the *first-order Born approximation*, usually referred to just as Born approximation.

This can also be put in a more compact form by writing the integration with Green's function as an operator:

$$\int G(\mathbf{r} - \mathbf{r}') [f(\mathbf{r}')] d\mathbf{r}' \equiv \hat{G}f$$

so (2.34) becomes $\psi = \psi_0 - \hat{G}V\psi$, and the series becomes

$$\begin{aligned}\psi^{(0)} &= \psi_i \\ \psi^{(1)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} \\ \psi^{(2)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \hat{G}V\hat{G}V\psi^{(0)} \\ &\dots \\ \psi^{(n)} &= \psi^{(0)} + \hat{G}V\psi^{(0)} + \dots + (\hat{G}V)^n\psi^{(0)}\end{aligned}$$

This form of the Born series helps visualising the structure of the n -th order approximation, and is the one usually found in quantum mechanics, for scattering of a wave on a potential V .

Naturally the (first-order) Born approximation is good only if the first correction is smaller than the incident field, and in general will be valid only if the series converges.

Note: in the Born approximation, if the wave is expressed as a sum of incident and diffracted secondary wave, the scattering of the secondary wave is neglected. So no multiple scattering.

2.3 Rytov Approximation

The Rytov approximation is obtained by representing the total field as a complex phase:

$$\psi(\mathbf{r}) = e^{\phi(\mathbf{r})} . \quad (2.35)$$

2.3 Rytov Approximation

Then, from the Helmholtz wave equation for ψ we have:

$$\nabla^2 e^{\phi(\mathbf{r})} + k^2 e^{\phi(\mathbf{r})} \quad (2.36)$$

Since

$$\nabla^2 e^{\phi(\mathbf{r})} = \nabla^2 \phi e^{\phi(\mathbf{r})} + (\nabla \phi)(\nabla \phi) e^{\phi(\mathbf{r})} ,$$

we get the following Riccati equation for the phase $\phi(\mathbf{r})$:

$$\nabla^2 \phi + (\nabla \phi)(\nabla \phi) + k^2 = 0 . \quad (2.37)$$

Let us now again write the refractive index as

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0 (1 + n_\delta(\mathbf{r})) . \quad (2.38)$$

The field for $n(\mathbf{r}) = 1$, i.e. the field in a non-refractive medium, can be written as $\psi_i(\mathbf{r}) = e^{\phi_i(\mathbf{r})}$; it is of course the incident field, and its phase will satisfy

$$\nabla^2 \phi_i + (\nabla \phi_i)^2 + k_0^2 = 0 \quad (2.39)$$

If we write $\phi = \phi_i + \phi_s$ and subtract (2.39) from (2.37), we get

$$\nabla^2 \phi_s + 2(\nabla \phi_i)(\nabla \phi_s) = -((\nabla \phi_s)(\nabla \phi_s) + k_0^2(n^2 - 1)) . \quad (2.40)$$

Now, using the identity

$$\nabla^2(\psi_i \phi_s) = (\nabla^2 \psi_i) \phi_s + 2\psi_i(\nabla \phi_i)(\nabla \phi_s) + \psi_i \nabla^2 \phi_s ,$$

equation (2.40) becomes:

$$\nabla^2(\psi_i \phi_s) + k^2 \psi_i \phi_s = ((\nabla \phi_s)(\nabla \phi_s) + k_0^2(n^2 - 1)) \psi_i , \quad (2.41)$$

whose solution can be written as an integral using the free-space Green's function, to give:

$$\phi_s(\mathbf{r}) = \frac{1}{\psi_i(\mathbf{r})} \int G(\mathbf{r} - \mathbf{r}') [(\nabla \phi_s(\mathbf{r}'))(\nabla \phi_s(\mathbf{r}')) + k_0^2(n^2(\mathbf{r}') - 1)] \psi_i(\mathbf{r}') d\mathbf{r}' \quad (2.42)$$

This equation is exact, but it's implicit and in practice provides no solution as it is. If we assume that the scattered phase ϕ_s is very small, then we can neglect $(\nabla \phi_s)^2$, and we obtain an approximate solution for the scattered phase

$$\phi_s(\mathbf{r}) \simeq \frac{1}{\psi_i(\mathbf{r})} \int G(\mathbf{r} - \mathbf{r}') [k_0^2(n^2(\mathbf{r}') - 1)] \psi_i(\mathbf{r}') d\mathbf{r}' \quad (2.43)$$

2.3 Rytov Approximation

The corresponding solution for the total field is then

$$\psi(\mathbf{r}) \simeq \psi_i(\mathbf{r})e^{\phi_s} . \quad (2.44)$$

This approximation is known as the (first) **Rytov approximation**. It corresponds to taking the first order term in an infinite power series expansion of the phase $\phi(\mathbf{r})$. It is valid when $(\nabla\phi_s)^2 \ll k_0^2(n^2(\mathbf{r}') - 1)$.

It is interesting to compare the validity of the Born and Rytov approximations.

Note that the Born approximation can be seen as a Taylor series approximation of the field $\psi(\mathbf{r}, \varepsilon)$ in powers of ε , where ε is a measure of the inhomogeneity. The Rytov approximation can also be seen as a Taylor series approximation of $\log \psi(\mathbf{r}, \varepsilon)$ in powers of ε . In our case, ε was the space-dependent variation n_δ from a constant refractive index.

[The following is **non-examinable**]. We shall reproduce here the analysis by Keller (see Keller J.B. 1969 'Accuracy and validity of the Born and Rytov approximations', J. Opt Soc. Am. 59, 1003-04) and consider the one-dimensional case of a wave travelling in an inhomogeneous medium given by

$$\psi(x, \varepsilon) = e^{ik(\varepsilon)x} , \quad (2.45)$$

and assume that $k(\varepsilon)$ is analytic in ε for $|\varepsilon|$ sufficiently small, so that it can be expanded in a power series in ε with coefficients $\mathbf{k}_j$:

$$k(\varepsilon) = \sum_{j=0}^{\infty} k_j \varepsilon^j . \quad (2.46)$$

The Born expansion gives

$$\psi(x, \varepsilon) = e^{ik_0 x} \sum_{s=0}^{\infty} \varepsilon^s \sum_{l=0}^s \frac{(ix)^l}{l!} \sum_{j_1+\dots+j_l=s} k_{j_1} \cdots k_{j_l} \quad (2.47)$$

The n th Born approximation $\psi_B^{(n)}(x, \varepsilon)$ is the sum of the first $n+1$ terms in the expression above:

$$\psi_B^{(n)}(x, \varepsilon) = e^{ik_0 x} \sum_{s=0}^n \varepsilon^s \sum_{l=0}^s \frac{(ix)^l}{l!} \sum_{j_1+\dots+j_l=s} k_{j_1} \cdots k_{j_l} \quad (2.48)$$

The Rytov expansion gives

$$\psi(x, \varepsilon) = e^{ik(\sum_{j=0}^{\infty} k_j \varepsilon^j)x} , \quad (2.49)$$

2.3 Rytov Approximation

and the n th Rytov approximation $\psi_R^{(n)}(x, \varepsilon)$ is obtained by taking the first $n + 1$ terms in the sum in the exponent:

$$\psi(x, \varepsilon) = e^{ik(\sum_{j=0}^n k_j \varepsilon^j)} . \quad (2.50)$$

The size of the error of the n th Born approximation $\psi - \psi_B^{(n)}$ for small ε and large $|x|$ can be found by examining the coefficient of ε^{n+1} in (2.47). That coefficient contains a term proportional to x^{n+1} . So

$$\psi - \psi_B^{(n)} = e^{ik_0 x} O(\varepsilon^{n+1} x^{n+1}) . \quad (2.51)$$

Dividing this by ψ , and noting that ψ differs from $e^{ik_0 x}$ by terms of the order ε , we obtain for the relative error:

$$\frac{\psi - \psi_B^{(n)}}{\psi} = O(\varepsilon^{n+1} x^{n+1}) . \quad (2.52)$$

The error for the n th Rytov approximation $\psi - \psi_R^{(n)}$ is:

$$\begin{aligned} \psi - \psi_R^{(n)} &= e^{ik(\sum_{j=0}^{\infty} k_j \varepsilon^j)} - e^{ik(\sum_{j=0}^n k_j \varepsilon^j)} \\ &= \psi \left(1 - e^{-ik(\sum_{j=n+1}^{\infty} k_j \varepsilon^j)} \right) \\ &= \psi O(\varepsilon^{n+1} x) \end{aligned}$$

Dividing this by ψ gives for the relative error

$$\frac{\psi - \psi_R^{(n)}}{\psi} = O(\varepsilon^{n+1} x) . \quad (2.53)$$

We can see then that the relative errors of the Born and the Rytov approximation are of the same order in the inhomogeneity parameter ε . However, the expressions obtained for the relative errors also show that they vary in a very different way as functions of x . For a single plane wave, the n th Rytov approximation is valid over a much larger range than is the n th Born approximation, however this advantage is lost for fields containing more than one wave, where the Rytov method must be applied to each wave separately and not to the total field ψ .

3 Scattering from randomly rough surfaces

3.1 Rayleigh criterion

The scattering of plane waves from a flat boundary between two media is a typical canonical problem, where analytical solutions are straightforward and well-known. It is an idealized case: all real surfaces are rough. The scattering problem will then depend on the 'roughness' of the surface, and exact analytical solutions will not be generally available. In this chapter we shall look at ways of characterizing the surface, and consider some approximate solutions.

Suppose then that a time-harmonic plane wave

$$\psi_i = \exp(ik[x \sin \theta - z \cos \theta])$$

is incident on a boundary which is now an irregular function of position. (We suppress above and in what follows the harmonic time dependence). We will assume here that the *surface normal* is well-defined and continuous everywhere along the boundary. One of the earliest treatments of the rough surface problem was by Rayleigh (1907), who considered the phase change due to height differences in the case when the wavelength is small compared with the horizontal scale of surface variation.

Calculating the phase difference $\Delta\phi$ between wavefronts along two specularly reflected rays as in the schematic diagram gives

$$\Delta\phi = 2k(h_2 - h_1) \cos \theta$$

where h_1, h_2 are the heights at the two points of incidence. The interference between these two rays depends on the magnitude of $\Delta\phi$ with respect to π . When the surface is nearly flat, $\Delta\phi \ll \pi$ and the two rays are in phase (so interfere constructively), but for large deviations we may have $\Delta\phi \sim \pi$, giving destructive interference. This leads to the so-called Rayleigh criterion for distinguishing different roughness scales, by which surfaces may be called 'rough' or 'smooth' according to whether $\Delta\phi$ greater than or less than $\pi/2$.

3.2 Surface Statistics

If this is averaged across the surface, then $(h_2 - h_1)$ may be replaced by the average r.m.s. surface height σ , which gives the surface r.m.s. deviation from a flat surface, and is defined by $\sigma^2 = \langle h^2(x) \rangle$. The Rayleigh criterion for 'smoothness' is then expressed by

$$k\sigma \cos \theta < \frac{\pi}{4}. \quad (3.1)$$

The quantity $k\sigma \cos \theta$ is referred to as the **Rayleigh parameter**. Note that this is dependent on angle of incidence, and implies that all surfaces become 'smooth' for low grazing angles. At optical wavelengths this is often reasonable, but is less true, for example, for typical radar wavelengths of 3cm or whenever the roughness length scale becomes comparable to a wavelength. In that case the Rayleigh criterion fails to take into account 'multiple scattering' effects such as shadowing and diffraction.

3.2 Surface Statistics

When we go on to the study of the Helmholtz integral equation, one of the main goals is to find dependence of *averaged* quantities on the statistics of the surface. We therefore require a few concepts and results for surface statistics and characterisation. (The necessary results are not extensive but some familiarity with them is essential in the manipulation of the statistical quantities which arise.)

Let S be a continuous irregular boundary, varying about a plane at, say, $z = 0$. We will assume that S can be represented as a function $h(x)$ of x , so that we can model this as a continuous stochastic process. We can think of h as a member of a given *ensemble* of surfaces all having the same statistical nature. All averages $\langle h(x) \rangle$ etc are averages over this ensemble. (The angled brackets denote ensemble averages.)

Main assumptions: A number of assumptions are usually made about the statistics of the rough surfaces. This is for analytical convenience, but in most cases the assumptions are physically reasonable.

(1) The mean surface is *flat*, i.e. $\langle h(x) \rangle = \text{constant}$ for all x (so we can choose $\langle h \rangle = 0$).

(2) The surface h is *statistically stationary in x* , i.e. all statistics are translationally invariant. Thus, in particular the autocorrelation function $\langle h(x)h(x + \xi) \rangle$ is a function of the spatial separation ξ only, and is constant in x .

(3) Surface heights are often assumed to be *normally distributed* (also referred to as Gaussian distributed, or simply as normal), i.e. they have

3.2 Surface Statistics

probability density function

$$f(h) = \frac{1}{\sigma\sqrt{2\pi}} e^{-h^2/2\sigma^2} . \quad (3.2)$$

For normal random variables we have the following:

If h is normal, then so is $h(x_1) + h(x_2)$, and $\int h(x)dx$ over any interval.

All the one-point statistics are determined by the mean $\langle h \rangle$ and variance $\langle h^2 \rangle$. For example we have $\langle h^{2n+1}(x) \rangle = 0$ for all n , and

$$\langle h^4(x) \rangle = 3\sigma^2 \langle h^2 \rangle \quad (3.3)$$

This can be seen by writing

$$\langle h^n \rangle = \int h^n f(h) dh$$

and integrating by parts, noting that in the case where $f(h)$ is Gaussian $hf(h) = -\sigma^2 \frac{d}{dh}(f(h))$.

The assumption of normal distributed heights is often physically reasonable; many rough surfaces arise as the result of a large number of independent random events and are therefore normal by the Central Limit Theorem. However, it is wrong for important cases such as the sea surface. (The sea typically has sharper peaks than troughs, so the height distribution is not symmetric about the mean, as would be required by the symmetry of the normal distribution about the origin.)

There are three main measures with which to characterise roughness:

- (1) **r.m.s. height** $\sigma = \sqrt{\langle h^2(x) \rangle}$ (since we assume $\langle h \rangle = 0$).
- (2) **Autocorrelation function** (a.c.f)

$$\rho(x_1, x_2) = \langle h(x_1)h(x_2) \rangle$$

By stationarity we can write this as a function of *spatial separation* only:

$$\rho(\xi) = \langle h(x)h(x + \xi) \rangle$$

(3) **Correlation length** L : This is defined as the value of separation ξ at which $\rho(\xi) = e^{-1}\rho(0)$. So large L corresponds to a slowly varying surface. Instead of L we often use the *mean slope*, $\langle |dh/dx| \rangle$. Clearly, slope scales with $1/L$.

3.2 Surface Statistics

The most general of these measures is clearly the a.c.f. (2), since this determines both the correlation length and r.m.s. height. It provides information about the spatial variation of the surface height, but is not related to the distribution of surface heights. The a.c.f. can have various forms depending on the nature of the irregularities.

Examples:

- (a) Gaussian a.c.f.: $\rho(\xi) = \sigma^2 e^{-\xi^2/L^2}$
- (b) Fractal surface: $\rho(\xi) = \sigma^2 e^{-|\xi|/L}$
- (c) Fourth order power law: $\rho(\xi) = \sigma^2(1 + |\xi|)e^{-|\xi|/a}$

Unlike (b), the functions (a) and (c) are smooth at the origin, i.e. $d\rho/d\xi = 0$ at $\xi = 0$. Thus ‘under a microscope’ a surface of this type would appear smooth. The autocorrelation function (c) often occurs in other contexts, such as turbulence. We can assume that ρ is an even function, and falls from its maximum σ at $\xi = 0$ to zero at large $|\xi|$.

We also need the **roughness spectrum** (or **power spectrum**), that is the Fourier transform of the a.c.f.:

$$S(\nu) = \int_{-\infty}^{\infty} \rho(\xi) e^{i\xi\nu} d\xi \quad (3.4)$$

All the definitions and observations about statistical quantities just given in the case of surface heights, also apply in the case of other random variables that are of interest to us, e.g. fluctuations in the dielectric constant ε_δ , or in the refractive index n_δ .

We note that 2-point correlations are not normally calculated directly. In practice, the ensemble average is often approximated by a time average over a finite data sample, for example:

$$\langle \varepsilon_\delta(x, t) \varepsilon_\delta(x + \xi, t) \rangle \simeq \frac{1}{T} \int_0^T \varepsilon_\delta(x_1, t) \varepsilon_\delta(x_2, t) dt$$

It is possible to do so only if we assume *stationary random process*. In this case, the **ergodic theorem** applies, which states that the ensemble average and the time average of a stationary random process must converge as the integration time goes to infinity:

$$\langle \varepsilon_\delta(x, t) \varepsilon_\delta(x + \xi, t) \rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \varepsilon_\delta(x, t) \varepsilon_\delta(x + \xi, t) dt \quad (3.5)$$

The equivalent result applies to infinite random surfaces:

$$\langle h(\mathbf{x}) h(\mathbf{x} + \xi) \rangle = \lim_{A_M \rightarrow \infty} \frac{1}{A_M} \int_{-\infty}^{\infty} h(\mathbf{x}) h(\mathbf{x} + \xi) d\mathbf{x} \quad (3.6)$$

3.2 Surface Statistics

If we use this expression for the a.c.f. in the power spectrum (3.4), we obtain

$$S(\nu) = \lim_{A_M \rightarrow \infty} \frac{1}{A_M} \left| \int_{-\infty}^{\infty} h(\mathbf{x}) e^{\nu \cdot \mathbf{x}} d\mathbf{x} \right|^2,$$

and therefore

$$\int_{-\infty}^{\infty} S(\nu) d(\nu) = \sigma^2.$$

The above is a special case of the general result that moments of the power spectrum give r.m.s. averages of higher order surface derivatives:

$$\int_{-\infty}^{\infty} S(\nu) \nu^{2n} d(\nu) = \left\langle \left(\frac{\partial^n h}{\partial x^n} \right)^2 \right\rangle \quad (3.7)$$

Finally in this section, the scattering solutions we seek are functions of the rough surface, involving integrals and derivatives of h . We therefore often need to evaluate the statistics of such functions, so we need some basic properties or rules for averaging.

(1) If $F(x)$ is a *deterministic* function, and $A(h)$ is any functional of the surface h , then

$$\left\langle \int A(h(x)) F(x) dx \right\rangle = \int \langle A(h(x)) \rangle F(x) dx$$

This follows by linearity of the integral.

(2) A function which sometimes arises is the average of the product of h and its slope:

$$\left\langle h(y) \frac{dh(x)}{dx} \right\rangle = \left. \frac{d\rho}{d\xi} \right|_{\xi=y-x}.$$

In order to prove (2), write

$$h(y)h'(x) = h(y) \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} [h(x + \epsilon) - h(x)]$$

The result follows by averaging the right-hand-side and taking the average inside the limit sign.

3.2 Surface Statistics

Numerical generation of random surfaces

It is instructive in the manipulation of averages to consider how a continuous rough surface $h(x)$ may be simulated. The simplest method is to represent $h(x)$ as a sum of sinusoidal components as follows:

Suppose we wish to represent an example of a surface with a given a.c.f. $\rho(\xi)$. The basic steps are:

- (1) Define $A(\nu) = \sqrt{B(\nu)}$ where B is the cosine transform of ρ ,

$$B(\nu) = \frac{2}{\pi} \int_{-\infty}^{\infty} \rho(\xi) \cos(\xi\nu) d\xi.$$

(We can assume that $B(\nu)$ has compact support.)

- (2) Choose some number N of equally-spaced frequencies $\nu_j = j\Delta\nu$, say, where N and ν_N are large enough to resolve the features of B adequately.

- (3) Choose N independent random phases ϕ_j , uniformly in $[0, 2\pi)$.

- (4) Define a function $h(x)$ by

$$h(x) = \sqrt{\Delta\nu} \sum_{n=1}^N A_n \sin(\nu_n x + \phi_n),$$

where $A_n = A(\nu_n)$. Then h is a continuous function of x with the required statistics, as we can show. The random part of this definition is in the choice of random phases (3). Each different set of phases gives rise to a new realisation of a random process h , and averages can therefore be taken over this ensemble.

First, it is easy to check that $\langle h \rangle = 0$, and for large N the values $h(x)$ are normally distributed by the central limit theorem. To calculate the a.c.f. of h , first write $x_n = \nu_n x + \phi_n$, and $y_n = \nu_n y + \phi_n$. Then since ϕ_n is uniform in $[0, 2\pi)$, it is easy to show for example that

$$\begin{aligned} \langle \sin x_n \rangle &= 0 \\ \langle \sin x_n \cos x_n \rangle &= 0 \\ \langle \sin^2 x_n \rangle &= 1/2 \\ \langle \sin x_n \sin y_n \rangle &= \frac{1}{2} \cos(\nu_n \xi) \end{aligned}$$

where $\xi = y - x$.

3.2 Surface Statistics

So the a.c.f. can be written

$$\begin{aligned}
 \langle h(x)h(y) \rangle &= \Delta\nu \sum_{m,n=1}^N A_m A_n \langle \sin x_n \sin y_m \rangle \\
 &= \Delta\nu \sum_{n=1}^N A_n^2 \langle \sin x_n \sin y_n \rangle \\
 &= \frac{\Delta\nu}{2} \sum_{n=1}^N A_n^2 \cos(\nu_n \xi) \\
 &\cong \int_{-\infty}^{\infty} B(\nu) \cos(\nu \xi) d\xi \\
 &= \rho(\xi)
 \end{aligned}$$

as required. Here we have used the fact that $\sin x_n$ and $\sin y_m$ are independent.

3.3 Properties and Approximate Solutions of Scattering Equations

3.3 Properties and Approximate Solutions of Scattering Equations

We will consider here the main methods used in solving the Helmholtz integral equations in the case of scattering from a rough surface, and the properties of the solutions.

Suppose that a plane wave

$$\psi_i(x, z) = e^{ik(x \sin \theta - z \cos \theta)}$$

impinges on a random rough surface $h(x)$. We will consider h to be a member of a statistical ensemble, which is stationary with respect to translation in x , with rms height $\langle h^2 \rangle = \sigma^2$, autocorrelation function $\rho(\xi)$. We usually require:

- the *scattered field* ψ_s ;
- the *coherent* (or mean) field $\langle \psi_s \rangle$;
- and the *field coherence function*

$$m(\xi) = \langle \psi_s(x) \psi_s^*(y) \rangle, \text{ where } \xi = y - x,$$

so that $m(0)$ is the mean intensity of the scattered field. It is often most important to find the *angular spectrum* $|\hat{\psi}(\nu)|^2$ or its average $\langle |\hat{\psi}(\nu)|^2 \rangle$, where

$$\hat{\psi}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_s(x, 0) e^{-i\nu x} dx \quad (3.8)$$

i.e. the Fourier transform of ψ_s along the horizontal mean plane, $z = 0$. Each Fourier component $\hat{\psi}(\nu)$ will be scattered away from the surface $z = 0$ as another plane wave

$$\hat{\psi}(\nu) e^{iqz}$$

satisfying the Helmholtz equation. This gives $q = \sqrt{k^2 - \nu^2}$, where we have taken the positive (or positive imaginary) root to ensure that the scattered field consists of outgoing waves.

The field at a point (x, z) in the medium can therefore be written

$$\psi_s(x, z) = \int_{-\infty}^{\infty} \hat{\psi}(\nu) e^{i(\nu x + qz)} d\nu \quad (3.9)$$

General properties:

We can state some general properties of these quantities.

(1) *Relation between $m(\xi)$ and $|\hat{\psi}(\nu)|^2$:*

3.3 Properties and Approximate Solutions of Scattering Equations

Consider the autocorrelation of $\hat{\psi}$. From (3.8) we obtain

$$\langle \hat{\psi}(\nu') \hat{\psi}^*(\nu) \rangle = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle \psi(x) \psi^*(y) \rangle e^{-i\nu x + i\nu' y} dx dy. \quad (3.10)$$

Make the changes of variables $\xi = (x - y)/2$, $Y = (x + y)/2$. This then becomes

$$\begin{aligned} \langle \hat{\psi}(\nu) \hat{\psi}^*(\nu') \rangle &= \frac{1}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m(2\xi) e^{-i(\nu+\nu')\xi - i(\nu-\nu')Y} d\xi dY \\ &= \frac{2}{\pi} \delta(\nu - \nu') \int_{-\infty}^{\infty} m(\xi) e^{-i\nu\xi} d\xi. \end{aligned} \quad (3.11)$$

This is just $2/\pi\delta(\nu - \nu')$ times the Fourier transform of $m(\xi)$. Notice the important corollary of this, that $\langle \hat{\psi}(\nu) \hat{\psi}^*(\nu') \rangle = 0$ for $\nu \neq \nu'$.

(2) *Energy conservation*, i.e. the average energy flux across a boundary in one direction must equal the average energy flux at the same point in the opposite direction. The averaged energy flux in a direction n is given by:

$$E(\psi, n) = -\frac{\rho\omega}{2} \text{Im} \left\{ \psi^* \frac{\partial \psi}{\partial n} \right\} \quad (3.12)$$

This is then integrated across the plane to which n is the normal to obtain the energy per unit area in the direction n . So for the homogeneous incident plane wave $\psi_i = \exp(ik[x \sin(\theta - z \cos(\theta))])$, the point-wise energy flux in the direction $\mathbf{n} = -z$, across some horizontal line is

$$E(\psi_i, n) = \frac{\rho\omega k \cos(\theta)}{2} \quad (3.13)$$

and for the scattered field (3.9), the average energy flux in the direction $n = z$ is

$$E(\psi_s(x, z)) = \frac{\rho\omega k}{2} \int_{-\infty}^{\infty} |\hat{\psi}(\nu)|^2 q d\nu \quad (3.14)$$

So energy conservation implies

$$\cos \theta = \int_{-\infty}^{\infty} q |\hat{\psi}(\nu)|^2 d\nu \quad (3.15)$$

where $q = \sqrt{k^2 - \nu^2}$.

3.3 Properties and Approximate Solutions of Scattering Equations

(3) *The mean field is specular, i.e.*

$$\langle \psi_s(x, z) \rangle = R_e(\theta) e^{ik[x \sin \theta + z \cos \theta]} \quad (3.16)$$

where the (generally unknown) constant R_e is an ‘effective reflection coefficient’ which depends on the angle and the surface statistics. This result is a generalised form of Snell’s law, and the mean transmitted field can be written similarly as a plane wave at the Snell’s law angle. (Correspondingly, the mean spectrum $\langle \hat{\psi} \rangle$ consists of a single delta-function peak.) The result follows from the assumption that the rough surface is statistically stationary. A corollary of this is that the mean of the full complex field shows no backscatter, or indeed any scatter outside the specular direction. This may initially be surprising, but note that it does *not* apply to the mean amplitude $\langle |\hat{\psi}| \rangle$ or the mean intensity or energy.

We now consider the two simplifying regimes of small surface height or small slope which allow approximate analytical solutions to be found.

(a) Small surface height $k\sigma \ll 1$:

In this case **perturbation theory** can be applied. The method is essentially to expand quantities appearing in the problem that are function of surface height, in order to form a simpler boundary problem on the mean plane, i.e. on $z = \langle h(x) \rangle = 0$.

We seek the solution for the scattered field ψ_s and its mean $\langle \psi_s \rangle$. Suppose that the surface obeys the Dirichlet condition, $\psi(x, h) = 0$. We proceed as follows:

(1) Expand the boundary condition to order h . Thus we obtain

$$\psi_i(x, 0) + \psi_s(x, 0) + h(x) \left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_s}{\partial z} \right) = 0 + O(h^2) \quad (3.17)$$

using $\psi = \psi_i + \psi_s$. Here and below, unless specified otherwise, the functions are to be evaluated on the mean plane $z = 0$.

(2) Next, assume that the scattered field everywhere can be expanded in powers of kh , say

$$\psi_s(x, z) = \psi_0(x, z) + \psi_1(x, z) + \psi_2(x, z) + \dots \quad (3.18)$$

where ψ_n is of order $O(h^n)$ for all n , so that ψ_0 is the known, deterministic *flat surface* reflected field, and ψ_n is stochastic for $n \geq 1$ since it depends on the specific choice of surface $h(x)$.

(3) Now truncate (3.18) at $O(h)$, substitute into (3.17), and neglect terms of order $O(h^2)$. This gives an *approximate boundary condition* which holds

3.3 Properties and Approximate Solutions of Scattering Equations

on the mean plane

$$\psi_i + \psi_0 + \psi_1 + h(x) \left(\frac{\partial \psi_i}{\partial z} + \frac{\partial \psi_0}{\partial z} \right) = 0 \quad (3.19)$$

where again all functions are evaluated at points $(x, 0)$. In this equation the third term ψ_1 is the only unknown component, since the remaining functions are the zero order (flat surface) forms, so we have an explicit approximation to the solution along the mean plane.

The first two terms in (3.19) cancel, since they represent the total field which would exist in the case of a flat surface, which vanishes by the Dirichlet boundary condition. We can now equate terms of equal order. Equating $O(h)$ (first order) terms gives

$$\psi_1 = -h(x) \left. \frac{\partial(\psi_i + \psi_0)}{\partial z} \right|_{(x,0)}$$

which gives

$$\psi_1(x, 0) = -2h(x) \frac{\partial \psi_i}{\partial z}. \quad (3.20)$$

This solves for ψ_1 explicitly on the mean plane. From this we can obtain the scattered field everywhere to $O(h)$, using $\psi_s = \psi_0 + \psi_1 + O(h^2)$. Once ψ_1 is known on any plane we can split it into Fourier components, and propagate these outwards (using radiation conditions to determine the direction):

Consider in particular the case of an incident plane wave, $\psi_i = e^{ik(x \sin \theta - z \cos \theta)}$. We then have

$$\psi_1(x, 0) = -2h(x)ik \cos \theta e^{ikx \sin \theta}. \quad (3.21)$$

Denote by $\hat{h}$ the transform of h ,

$$\hat{h}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} h(x) e^{-i\nu x} dx,$$

then, from (3.8) and (3.21) we get

$$\hat{\psi}(\nu) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \psi_s(x, 0) e^{-i\nu x} dx = -ik \frac{\cos \theta}{\pi} \hat{h}(\nu - k \sin \theta), \quad (3.22)$$

so that

$$\psi_1(x, z) = -ik \frac{\cos \theta}{\pi} \int_{-\infty}^{\infty} \hat{h}(\nu - k \sin \theta) e^{i(\nu x + qz)} d\nu \quad (3.23)$$

where as before $q = \sqrt{k^2 - \nu^2}$.

3.3 Properties and Approximate Solutions of Scattering Equations

We note that the formulation of the solution using perturbation theory in the approximation of small height depends on the boundary conditions. In particular, it will be different for Neumann and for impedance boundary conditions, although some results are applicable in general.

Averaging:

The dependence of the field on the surface is now clear to first order in surface height. Taking the average of (3.23) immediately gives the mean of this perturbation as

$$\langle \psi_1 \rangle = 0$$

everywhere, since $\langle h(x) \rangle = 0$, so that *first order perturbation theory predicts no change in the coherent field*. (Equivalently, the effective reflection coefficient is the same to first order as the flat surface coefficient.) Although we have examined the Dirichlet condition it holds for arbitrary boundary conditions since the first order term is always linear in the boundary itself.

Angular spectrum:

Now consider the angular spectrum to find the scattered energy. For a plane wave incident at angle θ on a given surface, the far-field intensity in the transform space is given by $I_\theta(\nu) = |\hat{\psi}(\nu)|^2$, so from (3.8), (3.21) we can write its average as

$$\langle |\hat{\psi}(\nu)|^2 \rangle = \left\langle \frac{k^2 \cos^2 \theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x) h(x') e^{i(k \sin \theta - \nu)(x - x')} dx' dx \right\rangle. \quad (3.24)$$

Making the change of variables $\xi = (x - x')$, $X = (x + x')$, this becomes

$$\begin{aligned} \langle |\hat{\psi}(\nu)|^2 \rangle &= \frac{k^2 \cos^2 \theta}{\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(\xi) e^{i(k \sin \theta - \nu)\xi} d\xi dX \\ &= 2 \frac{\delta(\nu)}{\pi} k^2 \cos^2 \theta S(k \sin \theta - \nu) \end{aligned} \quad (3.25)$$

where S is again the power spectrum of the surface and δ is the delta-function. Since the averaged scattered intensity is non-zero, this approximation to first order does lead to a contribution to the diffusely scattered field

$$\psi_d = \psi_{sc} - \langle \psi_{sc} \rangle$$

, even though it predicts no change in the coherent field, as we saw above. First order perturbation theory therefore does not obey conservation of energy.

3.3 Properties and Approximate Solutions of Scattering Equations

(b) Small surface slope:

We have been dealing with approximate solution in the case of small surface height. Now suppose that the *surface slopes are small*, i.e. $< |dh/dx| \gg 1$. We shall use this approximation with the integral form of the wave equation (1.87), so the scattered field at $\mathbf{r}$ is given by

$$\psi_{sc}(\mathbf{r}) = \int_S \psi(\mathbf{r}_0) \frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} - G(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi}{\partial n}(\mathbf{r}_0) d\mathbf{r}_0, \quad (3.26)$$

where $\mathbf{r}_0$ is on the surface and ψ and $\partial\psi/\partial n$ are unknown. We note that the use of this integral form implies integration over a closed surface, so will introduce errors (due to the edges) when the surface is not infinite.

The unknowns are approximated by using the **Kirchhoff approximation** (sometimes referred to as the tangent plane, or the geometrical optics solution), which treats any point on the scattering surface as though it were part of an infinite plane, parallel to the local surface tangent. We make the following assumptions:

- (1) that the surface can be treated as ‘locally flat’;
- (2) and that the incoming field at each point is just ψ_i .

The second assumption neglects multiple scattering, which can give rise to secondary illumination of any point on the surface.

Consider for simplicity the Dirichlet boundary condition, so that we are solving the integral equation

$$\psi_{sc}(\mathbf{r}_s) = - \int_S G(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi}{\partial n}(\mathbf{r}_0) d\mathbf{r}_0. \quad (3.27)$$

Under the assumptions above, we can approximate $\partial\psi/\partial n$ at each point by the value it would take for a flat surface with slope dh/dx :

$$\frac{\partial \psi}{\partial n} \cong -2 \frac{\partial \psi_i}{\partial n}. \quad (3.28)$$

This neglects curvature and shadowing by other parts of the surface. The field then becomes

$$\psi_s(\mathbf{r}) = 2 \int G(\mathbf{r}, \mathbf{r}_0) \frac{\partial \psi_i}{\partial n}(\mathbf{r}_0) d\mathbf{r}_0. \quad (3.29)$$

Similar formulae are easily obtained for Neumann condition and more generally an interface between two media.

When the surface is not perfectly reflecting, the normal derivative of the field at the surface will be given by

$$\frac{\partial \psi}{\partial n} \cong (1 - R(\mathbf{r}_0)) \frac{\partial \psi_i}{\partial n}, \quad (3.30)$$

3.3 Properties and Approximate Solutions of Scattering Equations

where $R(\mathbf{r}_0)$ is the flat surface reflection coefficient; and the field at the surface by:

$$\psi \cong (1 + R(\mathbf{r}_0))\psi_i . \quad (3.31)$$

If we further consider the *far-field approximation*, we can approximate the argument of the free space Green's function, $k|\mathbf{r} - \mathbf{r}_0|$ by

$$k|\mathbf{r} - \mathbf{r}_0| \cong kr - k\hat{\mathbf{r}} \cdot \mathbf{r}_0 , \quad (3.32)$$

where $\hat{\mathbf{r}}$ is the unit vector in the direction of observation $\mathbf{r}$. The derivative of the Green's function can then be approximated by

$$\frac{\partial G(\mathbf{r}, \mathbf{r}_0)}{\partial n} \cong -\frac{ie^{ikr}}{4\pi r} (\mathbf{n} \cdot \mathbf{k}_{sc}) e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0} , \quad (3.33)$$

where $\mathbf{k}_{sc} = k\hat{\mathbf{r}}$ is the wavevector of the scattered wave. Using these approximations in equation (3.26), we obtain for the scattered field

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r} \int_S ((R\mathbf{k}^- - \mathbf{k}^+) \cdot \mathbf{n}) e^{-i\mathbf{k}^- \cdot \mathbf{r}_0} d\mathbf{r}_0 , \quad (3.34)$$

where

$$\begin{aligned} \mathbf{k}^- &= \mathbf{k}_i - \mathbf{k}_{sc} \\ \mathbf{k}^+ &= \mathbf{k}_i + \mathbf{k}_{sc} . \end{aligned}$$

If θ_1 is the angle of incidence (measured from the normal), and θ_2 and θ_3 are, respectively, the angle of the scattered wave with the normal, and the angle of the scattered wave with the x -axis in the plane (x, y) , then

$$\begin{aligned} \mathbf{k}_i &= k(\hat{\mathbf{x}} \sin \theta_1 - \hat{\mathbf{z}} \cos \theta_1) \\ \mathbf{k}_{sc} &= k(\hat{\mathbf{x}} \sin \theta_2 \cos \theta_3 + \hat{\mathbf{y}} \sin \theta_2 \sin \theta_3 + \hat{\mathbf{z}} \cos \theta_2) . \end{aligned}$$

3.3 Properties and Approximate Solutions of Scattering Equations

We can now convert the integration in equation (3.34) to integration over the mean plane of the surface, S_M , by noting that an area element of the rough surface, $d\mathbf{r}_0$, projects onto the mean plane of an area element of the mean plane $d\mathbf{r}_M$, with the area elements related by

$$\mathbf{n}d\mathbf{r}_0 \cong \left(-\hat{\mathbf{x}}\frac{\partial h}{\partial x_0} - \hat{\mathbf{y}}\frac{\partial h}{\partial y_0} + \hat{\mathbf{z}} \right) d\mathbf{r}_M . \quad (3.35)$$

The scattered field can therefore be written in the general form

$$\psi_{sc}(\mathbf{r}) = \frac{ie^{ikr}}{4\pi r} \int_{S_M} \left(a\frac{\partial h}{\partial x_0} + b\frac{\partial h}{\partial y_0} - c \right) e^{ik(Ax_0 + By_0 + Ch(x_0, y_0))} dx_0 dy_0 , \quad (3.36)$$

where

$$\begin{aligned} A &= \sin \theta_1 - \sin \theta_2 \cos \theta_3 \\ B &= -\sin \theta_2 \sin \theta_3 \\ C &= -(\cos \theta_1 + \cos \theta_2) ; \end{aligned} \quad (3.37)$$

and

$$\begin{aligned} a &= \sin \theta_1(1 - R) + \sin \theta_2 \cos \theta_3(1 + R) \\ b &= \sin \theta_2 \sin \theta_3(1 + R) \\ c &= \cos \theta_2(1 + R) - \cos \theta_1(1 - R) . \end{aligned} \quad (3.38)$$

Note that this approximation for the scattered field has been derived within the far-field approximation, and for an incident plane wave. In order to make analytical manipulations possible, further approximations are usually made. In general, the reflection coefficient is a function of position on the surface. We shall assume instead that R is constant. With this approximation, and for $C \neq 0$, we can eliminate the terms involving partial derivatives of the surface by performing a partial integration. Carrying out the integration with the assumption of independent integration limits for x_0 and y_0 , and taking the surface to be of finite extent, defined by $-X \leq x_0 \leq X$ and $-Y \leq y_0 \leq Y$, gives a scattered field of the form

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r} 2F(\theta_1, \theta_2, \theta_3) \int_{S_M} e^{ik\phi(x_0, y_0)} dx_0 dy_0 + \psi_e , \quad (3.39)$$

where the phase function $\phi(x_0, y_0)$ is

$$\phi(x_0, y_0) = Ax_0 + By_0 + Ch(x_0, y_0) , \quad (3.40)$$

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the angular factor $F(\theta_1, \theta_2, \theta_3)$ is

$$F(\theta_1, \theta_2, \theta_3) = \frac{1}{2} \left(\frac{Aa}{C} + \frac{Bb}{C} + c \right) , \quad (3.41)$$

and the term ψ_e is given by

$$\begin{aligned} \psi_e(\mathbf{r}) = & -\frac{ie^{ikr}}{4\pi r} \left[\frac{ia}{kC} \int (e^{ik\phi(X, y_0)} - e^{ik\phi(-X, y_0)}) dy_0 \right. \\ & \left. + \frac{ib}{kC} \int (e^{ik\phi(x_0, Y)} - e^{ik\phi(x_0, -Y)}) dx_0 \right] \end{aligned} \quad (3.42)$$

In the above approximation the angular factor depends on the boundary conditions. The term ψ_e is often referred to as 'edge effects', since it involves the values of the phase function at the surface edges.

We can now calculate average quantities of the scattered field, when $h(x, y)$ is a random surface with some probability density $f(h)$. The average of the scattered field, i.e. the **coherent field** is given by

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r} 2F \int_{S_M} \int_{-\infty}^{\infty} e^{ik\phi(x_0, y_0)} f(h) dh dx_0 dy_0 . \quad (3.43)$$

Assuming stationarity, and using the explicit expression for the phase function given by equation (3.40), we obtain

$$\psi_{sc}(\mathbf{r}) = -\frac{ie^{ikr}}{4\pi r} 2F \hat{f}(kC) \int_{S_M} e^{ik(Ax_0 + By_0)} dx_0 dy_0 , \quad (3.44)$$

where $\hat{f}(kC)$ is the Fourier transform of the probability density function, with respect to the transform variable kC .

The average of the intensity, or of the **angular spectrum**, of the diffuse field $\psi_d = \psi_{sc} - \langle \psi_{sc} \rangle$ is given by

$$\langle |\psi_d|^2 \rangle = \langle \psi_{sc} \psi_{sc}^* \rangle - \langle \psi_{sc} \rangle \langle \psi_{sc}^* \rangle . \quad (3.45)$$

This expression is far more complicated than the equivalent one obtained in the 'small height' approximation, because the coherent field is now different from zero. Further approximations will be necessary to obtain an expression of practical use for the angular spectrum in the Kirchhoff approximation.

3.4 Depolarization of electromagnetic waves

3.4 Depolarization of electromagnetic waves

In Chapter 1 we saw that the electromagnetic scattering problem cannot in general be reduced to a scalar problem, and the Stratton-Chu equations are a system of coupled vector equations for the electric and magnetic field of a wave. It is therefore intuitive that a polarized electromagnetic wave with the electric field in a given plane, incident on an arbitrary surface, will give rise to a scattered wave with its electric field in a different plane.

In order to explore the effect of scattering from a surface on the polarization of a wave, we shall first define the geometry of the problem.

Let the surface S be defined by a function $h = h(x, y)$, with $z = 0$ as its mean level. Let bfE_i be the electric field of the incident wave, and bfE_{sc} be the electric field of the scattered wave, and we assume that the incident wave is linearly polarized.

The plane of incidence is the vertical plane containing $\mathbf{k}_I$, and corresponds in this case to the plane (x, z) . The scattering plane is the vertical plane containing $\mathbf{k}_{sc}$. The angle θ_1 is the angle between the z -axis and $\mathbf{k}_i$; θ_2 is the angle between the z -axis and $\mathbf{k}_{sc}$; and θ_3 is the angle between the x -axis and the scattering plane.

If $\mathbf{E}_i$ is in the plane of incidence, then we call the incident wave ***vertically polarized***.

If $\mathbf{E}_i$ is perpendicular to the plane of incidence, then we call the incident wave ***horizontally polarized***.

We shall denote by $\mathbf{e}^+$ and $\mathbf{e}^-$ unit vectors in the direction of vertical (+) and horizontal (−) polarization, so for example $\mathbf{e}_i^+ \perp \mathbf{e}_i^- \perp \mathbf{k}_i$, and a field $\mathbf{E}$ arbitrarily polarized can be decomposed into its vertical and horizontal components as $\mathbf{E} = E^+ \mathbf{e}^+ + E^- \mathbf{e}^-$.

When an incident field $\mathbf{E}_i$ is scattered by a surface S , the scattered field can be expressed in terms of the total field using Stratton-Chu equations:

$$\begin{aligned} \mathbf{E}_{sc}(\mathbf{r}) = & - \int_S [(\mathbf{n}(\mathbf{r}_0) \times \mathbf{E}(\mathbf{r}_0)) \times \nabla G(\mathbf{r}, \mathbf{r}_0) + (\mathbf{n}(\mathbf{r}_0) \cdot \mathbf{E}(\mathbf{r}_0)) \nabla G(\mathbf{r}, \mathbf{r}_0) \\ & + i\omega\mu(\mathbf{n}(\mathbf{r}_0) \times \mathbf{H}(\mathbf{r}_0))G(\mathbf{r}, \mathbf{r}_0)] dS . \end{aligned} \quad (3.46)$$

3.4 Depolarization of electromagnetic waves

It is usually impossible, though, to find exact expressions for the components of $\mathbf{E}_{sc}$ in the direction of horizontal and vertical polarization, and thus quantify exactly the degree of depolarization caused by the scattering.

We shall derive here approximate expressions for the components of $\mathbf{E}_{sc}$ by using a perturbation approximation over the mean plane $z = 0$, and in the special case of a perfectly conducting surface. In this case it is convenient to use a scalar Green's function $\bar{G}$ that is identically zero for $z = 0$ (i.e. satisfies a Dirichlet condition on the mean plane). This is achieved by choosing the half-space Green's function

$$\bar{G}(\mathbf{r}, \mathbf{R}) = \frac{e^{ik|\mathbf{r}-\mathbf{R}|}}{4\pi|\mathbf{r}-\mathbf{R}|} - \frac{e^{ik|\mathbf{r}-\mathbf{R}'|}}{4\pi|\mathbf{r}-\mathbf{R}'|}, \quad (3.47)$$

where $\mathbf{R}' = (x, y, -z)$ if $\mathbf{R} = (x, y, z)$.

We shall first make a far field approximation, so

$$\begin{aligned} \nabla \bar{G}(\mathbf{r}, \mathbf{r}_0) &\equiv \frac{\partial \bar{G}}{\partial x_0} \hat{\mathbf{x}} + \frac{\partial \bar{G}}{\partial y_0} \hat{\mathbf{y}} + \frac{\partial \bar{G}}{\partial z_0} \hat{\mathbf{z}} \\ &\simeq -2i\mathbf{k}_{sc} \frac{e^{ikr}}{4\pi r} e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0}, \end{aligned}$$

where $\mathbf{k}_{sc} = k\hat{\mathbf{r}}$. Then we use the far field approximation for the Green's function in (3.46) and take the surface integral onto the mean plane S_M to obtain:

$$\mathbf{E}_{sc}(\mathbf{r}) = -2i \frac{e^{ikr}}{4\pi r} \int_{S_M} [(\mathbf{n} \times \mathbf{E}) \times \mathbf{k}_{sc} + (\mathbf{n} \cdot \mathbf{E}) \mathbf{k}_{sc}] e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0} d\mathbf{r}_0. \quad (3.48)$$

But $\mathbf{E}_{sc}$ cannot have a component in $\mathbf{k}_{sc}$ direction, so the second term inside the above integral must be discarded and (3.48) reduces to:

$$\mathbf{E}_{sc}(\mathbf{r}) = 2i \frac{e^{ikr}}{4\pi r} \mathbf{k}_{sc} \times \int_{S_M} (\mathbf{n} \times \mathbf{E}) e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0} d\mathbf{r}_0. \quad (3.49)$$

Since we have assumed that the surface $S = h(x, y)$ is perfectly conducting, we have $\mathbf{E} \times \mathbf{n} = 0$ on S , and we can expand this to give boundary conditions on the mean surface S_M :

$$\mathbf{E} \times \mathbf{n}|_{z=h} = \mathbf{E} \times \mathbf{n}|_{z=0} + h \left(\frac{\partial (\mathbf{E} \times \mathbf{n})}{\partial z} \right) \Big|_{z=0} + \dots = 0 \quad (3.50)$$

If we now assume that we can expand $\mathbf{E}_{sc}$ as well:

$$\mathbf{E}_{sc} = \mathbf{E}_{sc}^{(0)} + \mathbf{E}_{sc}^{(1)} + \mathcal{O}(k^2 h^2) \quad (3.51)$$

3.4 Depolarization of electromagnetic waves

and recall that

$$\mathbf{n} \div \left(-\frac{\partial h}{\partial x}, -\frac{\partial h}{\partial y}, 1 \right),$$

so to order (0) $\mathbf{n}^{(0)} = (0, 0, 1)$, then matching terms of the same order in (3.50) and (3.4) (and using $\mathbf{E} = \mathbf{E}_i + \mathbf{E}_{sc}$) gives

to order (0):

$$\begin{aligned} E_{i_x} + E_{sc_x}^{(0)} &= 0, & E_{sc_x}^{(0)} &= -E_{i_x} \\ E_{i_y} + E_{sc_y}^{(0)} &= 0, & E_{sc_y}^{(0)} &= -E_{i_y} \end{aligned}$$

to order (1):

$$\begin{aligned} E_{sc_x}^{(1)} \Big|_{z=0} &= -2h \frac{\partial h}{\partial x} E_{i_z} \Big|_{z=0} - 2h \frac{\partial E_{i_x}}{\partial z} \Big|_{z=0} \\ E_{sc_y}^{(1)} \Big|_{z=0} &= -2h \frac{\partial h}{\partial y} E_{i_z} \Big|_{z=0} - 2h \frac{\partial E_{i_y}}{\partial z} \Big|_{z=0} \end{aligned}$$

To first order then (3.49) becomes:

$$\mathbf{E}_{sc}(\mathbf{r}) = i \frac{e^{ikr}}{2\pi r} \mathbf{k}_{sc} \times \int_{S_M} \left(\hat{\mathbf{x}} E_{sc_y}^{(1)} - \hat{\mathbf{y}} E_{sc_x}^{(1)} \right) e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0} d\mathbf{r}_0. \quad (3.52)$$

We can now use this expression to see whether, in this approximation, polarization is conserved or not, and quantify the scattered components.

Let us consider a linearly polarized incident plane wave. e.g. horizontally polarized, and, for simplicity, with unit amplitude:

$$\mathbf{E}_i = \hat{\mathbf{y}} e^{i\mathbf{k}_i \cdot \mathbf{r}}$$

If we use this incident field in the order (1) terms obtained above for the scattered field at the mean surface, and use the result in (3.52), we have:

$$\begin{aligned} \mathbf{E}_{sc}(\mathbf{r}) &= -i \frac{e^{ikr}}{\pi r} (\mathbf{k}_{sc} \times \hat{\mathbf{x}}) \int_{S_M} h(\mathbf{r}_0) \left(\frac{\partial E_{i_y}}{\partial z} \right) e^{-i\mathbf{k}_{sc} \cdot \mathbf{r}_0} d\mathbf{r}_0 \\ &= -i \frac{k^2 e^{ikr}}{\pi r} (\mathbf{k} \times \hat{\mathbf{x}}) \cos \theta_1 \int_{S_M} h(\mathbf{r}_0) e^{ik(Ax_0 + By_0)} dx_0 dy_0. \end{aligned}$$

We can now find the scattered horizontal and vertical polarization components by taking the dot product with the unit vectors $\mathbf{e}^-$ and $\mathbf{e}^+$ respectively, which in this geometry are given by:

$$\begin{aligned} \mathbf{e}^- &= \hat{\mathbf{x}} \sin \theta_3 + \hat{\mathbf{y}} \cos \theta_3 \\ \mathbf{e}^+ &= \hat{\mathbf{x}} \cos \theta_2 \cos \theta_3 + \hat{\mathbf{y}} \cos \theta_2 \sin \theta_3 - \hat{\mathbf{z}} \sin \theta_2 \end{aligned}$$

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and we obtain the result that in this approximation the far field scattered intensity in the β polarization direction obtained from an incident field with α polarization is in general different from zero, which we shall write as: $\langle I_{\alpha \rightarrow \beta}^{(1)} \rangle \neq 0$ (here α and β denote either vertical and horizontal polarization). For an incident wave with components in both polarization directions, the general result can be expressed as:

$$\langle I_{\alpha \rightarrow \beta}^{(1)} \rangle = \frac{k^4}{r^2} \Phi_{\alpha \rightarrow \beta}(\theta_1, \theta_2, \theta_3) A_M \left| \int_{S_M} \langle h(\mathbf{r}_0), h(\mathbf{r}_0 + \mathbf{r}'_0) \rangle e^{ik(Ax'_0 + By'_0)} dx'_0 dy'_0 \right|^2 \quad (3.53)$$

where $\Phi_{\alpha \rightarrow \beta}$ is an angular factor dependent on the polarization of the incident and scattered waves, A_M is the area of the mean surface, and the integral is the power spectrum of the surface (recall definition of power spectrum of a surface in section 3.3).

For the particular case of a horizontally polarized incident wave, the angular factors obtained are:

$$\begin{aligned} \Phi_{H \rightarrow H} &= 4 \cos^2 \theta_1 \cos^2 \theta_2 \cos^2 \theta_3, \\ \Phi_{H \rightarrow V} &= 4 \cos^2 \theta_1 \sin^2 \theta_3 \end{aligned} \quad (3.54)$$

For the particular case of a vertically polarized incident wave, the angular factors are:

$$\begin{aligned} \Phi_{V \rightarrow V} &= 4(\sin \theta_1 \sin \theta_2 - \cos \theta_3)^2, \\ \Phi_{V \rightarrow H} &= 4 \cos^2 \theta_2 \sin^2 \theta_3 \end{aligned} \quad (3.55)$$

There will be some angles for which some of these terms vanish. In particular, if $\theta_3 = 0$, i.e. for scattering in the azimuthal plane, both cross-polarization terms are zero: $\Phi_{H \rightarrow V} = \Phi_{V \rightarrow H} = 0$, therefore

$$\langle I_{H \rightarrow V}^{(1)} \rangle = \langle I_{V \rightarrow H}^{(1)} \rangle = 0$$

and first order perturbation theory predicts no depolarization in the azimuthal plane. We know from experimental results and numerical simulations that depolarization does occur in the azimuthal plane. The result obtained here shows that it is a higher order effect.

4 Wave Propagation through Random Media

References:

- A. Ishimaru, Wave Propagation and Scattering in Random Media
 B.J. Uscinski, Elements of Wave Propagation in Random Media

Remarks

This section concerns waves scattered by randomness or irregularities in the medium through which they are propagating. In many situations the wave speed varies randomly, for example in the atmosphere or the ocean. Sometimes this variation may be highly localized, such as a patch of turbulent air (e.g. over a hot road) or bathroom glass. These effects cause *focusing*, somewhat like that of a lens, and produce regions of both high and low intensity. (Familiar examples include the twinkling of stars, or the pattern of light in a swimming pool.)

There are essentially two mechanisms which contribute to these effects, and we shall refer to them as:

- (i) *diffraction* (distance effect): i.e. the evolution of an irregular wave beyond a fixed plane. This allows focusing of rays as in a lens even when the medium is homogeneous; and
- (ii) *scattering*, i.e. the continuous evolution of phase with propagation due to extended irregularities, causing bending of rays.

We will consider these mechanisms only for *weakly scattering media*. Roughly speaking, ‘weak scattering’ corresponds to small angles of scatter, so that a plane wave may become scattered into a narrow range of directions close to the original direction. This allows us to use the *parabolic wave equation*, which was derived in Lecture 3 as a small angle approximation. We will assume throughout this section that the parabolic equation holds, and that there is a definite predominant direction of propagation (which can be taken to be horizontal).

In an extended medium the effects (i) and (ii) mentioned above of course occur simultaneously, but we shall see that it is possible to treat them separately under reasonable assumptions.

We shall first consider the case in which the random irregularities occur within a thin layer.

4.1 Propagation beyond a thin phase screen

Suppose that we have initially a plane wave $\psi = e^{ikx}$ of unit amplitude propagating horizontally, so that the *reduced wave*, i.e. ψe^{-ikx} , is just $E(x, z) \equiv 1$.

4.1 Propagation beyond a thin phase screen

Suppose that E encounters a thin vertical layer in the region $x \in [-\xi, 0]$, say, in which the wave speed $c(z)$ is slightly irregular. (This may represent for example a jet of hot air, or a turbulent layer.)

Denote the **refractive index** $n(z) = c_0/c(z)$, where c_0 is the background or free wave speed, and write

$$n(z) = 1 + w(z) . \quad (4.1)$$

We will assume that

- the function $w(z)$ is small: $w(z) \ll 1$,
- $w(z)$ is a continuous random fluctuation, with the following properties:
 - it has mean zero, i.e. $\langle w(z) \rangle = 0$ for all z ,
 - it is stationary in z , so, e.g. $\langle w(z_1)w(z_2) \rangle = \langle w(z)w(z + \xi) \rangle$,
 - it is normally distributed, i.e. its probability distribution function is Gaussian.

Initial effect: In the assumption of weak scattering and for a thin enough layer, the field will only suffer a phase change on going through the layer. If a wave has wavenumber k before entering the layer, the wavenumber in the layer will be given by $kn(z) = k + kw(z)$, and the reduced wave will acquire a phase

$$\phi(z) = k\xi w(z), \quad (4.2)$$

where ξ is the thickness of the layer.

Then E emerges from the layer with a pure phase change,

$$E(0, z) = e^{i\phi(z)} \quad (4.3)$$

4.1 Propagation beyond a thin phase screen

Evolution of the field and the moment equations

We shall use the thin screen model described above, and the parabolic equation which applies under the assumptions we made, to derive evolution equations for the **moments of the field**, in particular the **first moment** or mean field

$$m_1(x) = \langle E(x, z) \rangle. \quad (4.4)$$

(note this is a function of x only, by stationarity)

and the **second moment** (transverse autocorrelation) of the field, defined as

$$m_2(x, \eta) = \langle E(x, z) E^*(x, z + \eta) \rangle \quad (4.5)$$

so that the mean of the intensity $I(x, z) = |E|^2$ can be written $\langle I(x) \rangle = m_2(x, 0)$.

Evolution equations will be derived first just for propagation beyond a thin screen, as an introduction to the concept, then for propagation in an extended random medium, for some of the moments of the field: the *moment equations*.

It is primary aim of the study of random media, to examine the evolution of the field E with distance beyond a layer and find its statistics. There are many reasons for this requirement: for example in ocean acoustics one can almost never know the refractive index in detail, but statistical information can help overcome communications and navigational problems, or may be used for remote sensing of the environment. In other situations the measurement devices themselves may be detecting time or spatial averages.

Suppose for example we wish to find the mean intensity of the field. For a given medium it will not be possible to obtain a general solution for the wavefield or its intensity as a function of position. However, some of the statistical moments, such as field autocorrelation, themselves obey evolution equations which take a relatively simple form since the fluctuations in the medium have been ‘averaged out’, and can be solved or their solutions approximated analytically.

Before deriving equations for the evolution of the first and second moments, we shall make some heuristic remarks.

As the field evolves, the pure phase fluctuations which are imposed initially, equation (4.3), become converted to amplitude variations. (In terms of ray theory, this happens as the layer focuses or de-focuses the rays passing through it, and the intensity changes with the ray density.)

This can be shown and quantified roughly as follows:

At a small distance x beyond the layer, we can take a Taylor expansion of the field $E(x, z)$ about $E(0, z) = e^{i\phi(x)}$. If we then use (4.3) and the parabolic

4.1 Propagation beyond a thin phase screen

wave equation

$$\frac{\partial E}{\partial x} = \frac{i}{2k} \frac{\partial^2 E}{\partial z^2}. \quad (4.6)$$

we obtain

$$E(x, z) \cong \left[1 + \frac{i}{2k} x (i\phi'' - \phi'^2) \right] e^{i\phi}, \quad (4.7)$$

where the prime denotes derivative, $\phi' = d\phi/dz$ etc., so that

$$I(x, z) \cong 1 - \frac{x}{k} \phi'' + \frac{x^2}{4k^2} (\phi''^2 + \phi'^4) \quad (4.8)$$

neglecting higher powers of x . This describes the initial mechanism for the build-up of amplitude fluctuations across the wavefront.

However, we can form *evolution equations*, i.e. differential equations governing the behaviour of the moments. These can be solved to find the far-field. The first few moment equations are trivial in the case of propagation beyond a layer, but are a useful introduction to the moment equations, and illustrate simply some important concepts..

Evolution of the first moment (mean field):

We note first that, since $E(x, z) = 1$ before the screen and $E(0, z) = \exp(i\phi(z))$ immediately after the screen, the initial intensity is unchanged: $\langle I(0, z) \rangle \equiv 1$, and the initial mean field is

$$m_1(0) = \langle e^{i\phi(z)} \rangle = e^{-\sigma^2/2}. \quad (4.9)$$

This is exact for the normal distribution as assumed here, in which case the the probability density function of ϕ is

$$f(\phi) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\phi^2/2\sigma^2}.$$

and approximate in general. It can be obtained from the definition

$$\langle e^{i\phi} \rangle = \int_{-\infty}^{\infty} e^{i\phi} f(\phi) d\phi, \quad (4.10)$$

or simply by expanding the exponential and averaging term by term,

$$\langle e^{i\phi} \rangle = 1 + i \langle \phi \rangle - \frac{\langle \phi^2 \rangle}{2} - i \frac{\langle \phi^3 \rangle}{3} + \dots \quad (4.11)$$

By equation (4.9) we can write

$$\langle \hat{E}(0, \nu) \rangle = \int_{-\infty}^{\infty} m_1(0) e^{i\nu z} dz = \sqrt{2\pi} \delta(\nu) e^{-\sigma^2/2}. \quad (4.12)$$

4.1 Propagation beyond a thin phase screen

But $\hat{E}$ satisfies the parabolic equation, so

$$\hat{E}(x, \nu) = e^{-i\frac{\nu^2}{2k}x} \hat{E}(0, \nu) \quad (4.13)$$

Taking the average of (4.13) and using (4.12) then gives

$$\langle \hat{E}(x, \nu) \rangle = \sqrt{2\pi} \delta(\nu) e^{-i\nu^2 x/2k} e^{-\sigma^2/2},$$

so that (because of the delta function) $\langle \hat{E}(x, \nu) \rangle = \langle \hat{E}(0, \nu) \rangle$ for all z , i.e.

$$\frac{dm_1}{dx} = 0. \quad (4.14)$$

The mean field is unchanged with distance.

Evolution of the second moment (vertical correlation of field):

We are also interested in mean intensity $\langle I(x) \rangle$. Although we cannot form an evolution equation for $\langle I(x) \rangle$ itself, we can do so for $m_2(x, \eta)$ (equation (4.5)) and obtain $\langle I \rangle$ by solving and setting $\eta = 0$.

The *initial condition* for m_2 at $x = 0$, just beyond the screen, is given by

$$m_2(0, \eta) = \langle e^{i[\phi(z_1) - \phi(z_2)]} \rangle$$

where $\eta = z_1 - z_2$. Since ϕ is normally distributed, so is the difference $\phi(z_1) - \phi(z_2)$. The variance of this difference is

$$\langle [\phi(z_1) - \phi(z_2)]^2 \rangle = 2 [\sigma^2 - \rho(\eta)],$$

where we have denoted by $\rho(\eta)$ the transverse autocorrelation of the layer ϕ :

$$\rho(\eta) = \langle \phi(z) \phi(z + \eta) \rangle \quad (4.15)$$

with variance

$$\sigma^2 = \rho(0). \quad (4.16)$$

This gives the initial value

$$m_2(0, \eta) = e^{-[\sigma^2 - \rho(\eta)]}. \quad (4.17)$$

Now consider the ‘transform’ moment M_2 defined by

$$M_2(x, \nu_1, \nu_2) = \langle \hat{E}(x, \nu_1) \hat{E}^*(x, \nu_2) \rangle.$$

By using the parabolic equation (4.6) for the transform field we get

$$\frac{\partial M_2}{\partial x} = \frac{i}{2k} (\nu_2^2 - \nu_1^2) M_2 \quad (4.18)$$

4.2 Propagation in an extended random medium

However we can write M_2 directly in terms of E , as

$$\begin{aligned} M_2(x, \nu_1, \nu_2) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle E(x, z_1) E^*(x, z_2) \rangle e^{i(\nu_1 z_1 - \nu_2 z_2)} dz_1 dz_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} m_2(x, \eta) e^{i(\nu_1 - \nu_2)Y/2 + i(\nu_1 + \nu_2)\eta/2} d\eta dY \end{aligned} \quad (4.19)$$

where we have made the change of variables $\eta = z_1 - z_2$, $Y = z_1 + z_2$. Evaluating the Y -integral in (4.19) gives

$$M_2(x, \nu_1, \nu_2) = \sqrt{2\pi} \delta(\nu_1 - \nu_2) \int_{-\infty}^{\infty} m_2(x, \eta) e^{i(\nu_1 + \nu_2)\eta/2} d\eta \quad (4.20)$$

so that M_2 vanishes unless $\nu_1 = \nu_2$. Hence we see from equation (4.18) that M_2 , and therefore m_2 , does not evolve with x , i.e.

$$\frac{\partial M_2}{\partial x} = \frac{\partial m_2}{\partial x} = 0. \quad (4.21)$$

In particular the mean intensity remains constant. (It will be seen later that this no longer holds for an extended random medium.) We therefore need to go to higher moments to describe the intensity fluctuations which the eye and most ‘square law’ detectors observe in waves propagating through an irregular layer. Before doing that, we shall consider the evolution of the first and second moments in an extended random medium.

4.2 Propagation in an extended random medium

Consider now the second mechanism which can produce field fluctuations, that of extended refractive index irregularities. This is common in many situations, e.g. underwater acoustic, or atmospheric radio wave propagation. (Apart from any random irregularities there is often an underlying profile; for example the ocean sound channel which causes upward refraction of ray paths, confining sound to a region near the surface. This will not be treated here.)

Consider again a 2-dimensional medium (x, z) and a time-harmonic wave $\phi e^{i\omega t}$. Let $c(x, z)$ be the wave speed in the medium, and c_0 be the ‘reference’

4.2 Propagation in an extended random medium

or average wave speed. (We will take this as constant here although the actual profile may depend on depth.) Let $k = \omega/c_0$ be the corresponding wavenumber.

Denote the **refractive index** by $n(x, z) = c_0/c(x, z)$. We can write

$$n = 1 + n_d(z) + \mu W(x, z) \quad (4.22)$$

where n_d is the deterministic profile which, for example, allows for channelling, but which will be set to zero in the following derivation. μW is the random part, where W has been normalised, so that

$$\langle W \rangle = 0, \quad \langle W^2 \rangle = 1,$$

and therefore μ^2 is the variance of n . We will take W to be normally distributed, and *stationary in x and z* . We can then define the 2-dimensional autocorrelation function

$$\rho(\xi, \eta) = \mu^2 \langle W(x, z) W(x + \xi, z + \eta) \rangle \quad (4.23)$$

so that $\rho(0, 0) = \mu^2$. **Note** that ρ is assumed to decay to zero as $\xi \rightarrow \infty$ or $\eta \rightarrow \infty$. (This is reasonable unless there is an underlying periodicity in the medium.)

Further define the *horizontal and vertical length scales* H, L defined by

$$\rho(H, 0) = \rho(0, L) = \mu^2 e^{-1}.$$

There are thus at least three measures affecting the scattering in different ways: μ^2 , H , and L . We will look at their various effects on the field.

Weak scatter assumptions: We make the following assumptions, which correspond to different forms of weak scattering restrictions.

(1) *Small variation of refractive index*, $\mu^2 \ll 1$ (or equivalently $|n^2 - 1| \ll 1$).

(2) *Small angles of scatter*, expressed as

$$\lambda_0 \ll L$$

where λ_0 is the reference wavelength, $\lambda_0 = 2\pi/k_0$.

(3) *Weakly scattering medium*, i.e. the phase fluctuations imposed over a distance H are small,

$$k_0 \mu H \ll 1$$

4.2 Propagation in an extended random medium

Note: It will be seen below that ‘stretching’ the scale size H increases the scattering effect, whereas stretching the vertical scale L weakens it.

Under these weak scatter assumptions we will be able to use the parabolic equation for an extended random medium, which was derived in section 3.4:

$$\frac{\partial E}{\partial x} = \frac{i}{2k} \frac{\partial^2 E}{\partial z^2} + \frac{ik}{2} (n^2(x) - 1)E. \quad (4.24)$$

Moment equations for an extended random medium

We now go on to formulating and solving equations for the evolution of the moments, analogous to those for the thin layer. In the derivations that follow, we shall consider separately the ‘scattering effect’ and the ‘diffraction effect’, i.e. the two terms on the r.h.s. of the parabolic equation (4.24).

Define again the **first moment**

$$m_1(x) = \langle E(x, z) \rangle \quad (4.25)$$

where this quantity is again independent of z by the stationarity of W . Thus all z -derivatives $d^n m_1/dz^n$ vanish, so that *all effects on the mean field are due to the scattering term only* (in eq. (4.24)).

In order to derive the first moment equation, consider first the phase change $\phi(z)$ over a distance $d > H$ due to the scattering term only:

$$E(x + d, z) = E(x, z) e^{i\phi(z)} \quad (4.26)$$

where

$$\phi(z) = k_0 \mu \int_x^{x+d} W(x', z) dx'. \quad (4.27)$$

Averaging (4.26) will involve a term $\langle \phi^2 \rangle$, so square and average (4.27) to get

$$\begin{aligned} \langle \phi^2 \rangle &= k_0^2 \mu^2 \int_x^{x+d} \int_x^{x+d} \langle W(x', z) W(x'', z) \rangle dx' dx'' \\ &= k_0^2 \mu^2 \int_x^{x+d} \int_x^{x+d} \rho(x' - x'', 0) dx' dx'', \end{aligned}$$

where we have used the definition (4.23) for the transverse autocorrelation $\rho(x' - x'', 0)$. We now make the change of variables

$$\begin{aligned} \xi &= x' - x'' \\ X &= (x' + x'')/2 \end{aligned}$$

4.2 Propagation in an extended random medium

and use $d > H$ together with the fact that $\rho(\xi, 0) \sim 0$ for large ξ to obtain

$$\langle \phi^2 \rangle \cong k^2 \mu^2 \int_0^d \int_{-\infty}^{\infty} \rho(\xi, 0) d\xi dX.$$

Therefore

$$\langle \phi^2 \rangle = k^2 \mu^2 \sigma_0 d \quad (4.28)$$

where

$$\sigma_0 = \int_{-\infty}^{\infty} \rho(\xi, 0) d\xi.$$

Now, averaging (4.26) and using (4.28) gives

$$m_1(x+d) \equiv \langle E(x+d, z) \rangle \cong m_1(x) e^{-k_0^2 \mu^2 \sigma_0 / 2d} \quad (4.29)$$

where we have made a further key assumption: *the field becomes independent of the medium*, due to the cumulative effect of scattering., i.e. for large x

$$\langle E(x, z) e^{i\phi(z)} \rangle \sim \langle E(x, z) \rangle \langle e^{i\phi(z)} \rangle.$$

This is because the moments of E at any distance in the medium are the result of the cumulative effective of the medium - of scattering - up to that distance, however the statistics of the medium in a layer of thickness d are independent of the statistics of the medium elsewhere.

It now follows directly from (4.29) that

$$m_1(x) = e^{(-k_0^2 \mu^2 \sigma_0 / 2)x} m_1(0). \quad (4.30)$$

Equivalently (or expanding $m_1(x+\xi)$ in ξ and comparing terms of $O(\xi)$ with a Taylor series) we can write

$$\frac{dm_1}{dx} = -\left(\frac{1}{2} k_0^2 \mu^2 \sigma_0\right) m_1. \quad (4.31)$$

Thus $m_1(x)$ decays exponentially and is purely real.

4.2 Propagation in an extended random medium

Equation (4.31) for the evolution of the first moment due to the scattering term only, is also valid for a more general incident wave in 3 dimension, and amplitude different from 1, where the wave emerging from from a screen of width d is given by:

$$E(x + \xi, y, z) = E(x, y, z)e^{\phi(x+\xi, y, z)} ,$$

and we have

$$\frac{\partial m_1}{\partial x} = -(\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 . \quad (4.32)$$

In general, we cannot disregard the ‘diffraction’ term, and we need to use

$$\frac{\partial E}{\partial x} = \frac{i}{2k_0} \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) E + \frac{ik_0}{2}(n^2 - 1)E . \quad (4.33)$$

Therefore the **equation for the first moment** is

$$\frac{\partial m_1}{\partial x} = \frac{i}{2k_0} \left(\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial y^2} \right) m_1 - (\frac{1}{2}k_0^2\mu^2\sigma_0)m_1 . \quad (4.34)$$

As before, since the medium is stationary, and therefore m_1 is independent of the transverse directions y, z , this reverts to equation (4.32) above.

For higher moments, we shall see that the ∇^2 term must be retained. The evolution equations could be solved by applying some small perturbations method, for example Born or Rytov, but only for small intensity fluctuations. Such solutions are of very limited use, since we know from experimental results and observations that even small randomness can give rise to very large intensity fluctuations.

It is possible to find a solution that allows for large intensity fluctuations by a *local* application of the method of small perturbation, and we shall derive moment equations and their solutions in this way. Conceptually then, using these moment equations to describe the evolution of the field is equivalent to using repeated applications of the Born approximation for successive (thin) screens.

Let us now consider the **second moment**

$$m_2 = \langle E_1(x, y_1, z_1)E_2^*(x, y_2, z_2) \rangle , \quad (4.35)$$

where E_1 and E_2 represent E at two separate points in the same transverse plane at x .

4.2 Propagation in an extended random medium

Let us derive first the ‘diffraction’ term (or ‘distance effect’). Consider

$$\frac{\partial}{\partial x} E_1 E_2^* = E_2^* \frac{\partial E_1}{\partial x} + E_1^* \frac{\partial E_2}{\partial x} .$$

The diffraction term for the field at a single point E_j (where $j = 1, 2$) is

$$\frac{\partial E_j}{\partial x} = \frac{i}{2k_0} \left(\frac{\partial^2}{\partial y_j^2} + \frac{\partial^2}{\partial z_j^2} \right) E_j \equiv -\frac{i}{2k_0} \nabla_{Tj}^2 E_j . \quad (4.36)$$

Therefore

$$\frac{\partial}{\partial x} E_1 E_2^* = \frac{i}{2k_0} (E_2^* \nabla_{T1}^2 E_1 - E_1 \nabla_{T2}^2 E_2^*) ,$$

and taking the ensemble average

$$\frac{\partial}{\partial x} \langle E_1 E_2^* \rangle = \frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T2}^2) \langle E_1 E_2^* \rangle . \quad (4.37)$$

We shall now consider the ‘scattering’ effect due to a screen of thickness d , so how the second moment $\langle E_1 E_2^*(x) \rangle$ evolves onto $\langle E_1 E_2^*(x+d) \rangle$. We have:

$$E_1 E_2^*(x+d) = E_1 E_2^*(x) e^{i[\phi(x+d, y_1, z_1) - \phi(x+d, y_2, z_2)]} \quad (4.38)$$

and

$$E_1 E_2^*(x+d) \simeq E_1 E_2^*(x) + \frac{\partial}{\partial x} (E_1 E_2^*) d , \quad (4.39)$$

so

$$E_1 E_2^*(x+d) = E_1 E_2^*(x) [1 + i(\phi(y_1, z_1) - \phi(y_2, z_2)) - \frac{1}{2}(\phi(y_1, z_1) - \phi(y_2, z_2))^2 + \dots] , \quad (4.40)$$

where we have expanded the exponent in (4.38). If we now equate (4.38) and (4.39), by taking the ensemble average, and remembering that $\langle \phi \rangle = 0$, we have

$$\frac{\partial}{\partial x} \langle E_1 E_2^* \rangle d = -\frac{1}{2} \langle (\phi(y_1, z_1) - \phi(y_2, z_2))^2 \rangle \langle E_1 E_2^* \rangle . \quad (4.41)$$

Now consider

$$\langle (\phi_1 - \phi_2)^2 \rangle = (\langle \phi_1^2 \rangle - 2 \langle \phi_1 \phi_2 \rangle + \langle \phi_2^2 \rangle) , \quad (4.42)$$

where $\phi_i = \phi(y_i, z_i)$. In the same way as we previously derived $\langle \phi^2 \rangle$, (equation (4.28)), we can derive

$$\langle \phi_1 \phi_2 \rangle \cong k_0^2 \mu^2 \int_0^d \int_{-\infty}^{\infty} \rho(\xi, 0) d\xi dX .$$

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Therefore

$$\langle \phi_1 \phi_2 \rangle = k_0^2 \mu^2 d\sigma(\eta, \zeta) \quad (4.43)$$

where

$$\sigma(\eta, \zeta) = \int_{-\infty}^{\infty} \rho(\xi, \eta, \zeta) d\xi,$$

$\eta = y_1 - y_2$, $\zeta = z_1 - z_2$, and ρ is the normalised autocorrelation function of the refractive index fluctuation:

$$\rho(\xi, \eta, \zeta) = \frac{1}{\mu^2} \langle W(x_1, y_1, z_1) W(x_2, y_2, z_2) \rangle.$$

We can now use (4.43) in (4.42) to obtain

$$\langle (\phi_1 - \phi_2)^2 \rangle = 2k_0^2 \mu^2 d(\sigma(0, 0) - \sigma(\eta, \zeta)). \quad (4.44)$$

Therefore the evolution due to the scattering only is

$$\frac{\partial}{\partial x} \langle E_1 E_2^* \rangle = -k_0^2 \mu^2 (\sigma(0, 0) - \sigma(\eta, \zeta)) \langle E_1 E_2^* \rangle \quad (4.45)$$

Now, combining (4.45) and (4.37), we obtain the second moment equation

$$\frac{\partial m_2}{\partial x} = \frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T2}^2) m_2 - k_0^2 \mu^2 (\sigma(0, 0) - \sigma(\eta, \zeta)) m_2. \quad (4.46)$$

Let us consider the **fourth moment** defined by

$$m_4 = \langle E_1 E_2^* E_3 E_4^* \rangle. \quad (4.47)$$

We can derive an equation for the fourth moment in the same way as before:
The 'distance' effect:

$$\frac{\partial}{\partial x} \langle E_1 E_2^* E_3 E_4^* \rangle = \frac{\partial E_1}{\partial x} \langle E_2^* E_3 E_4^* \rangle + E_1 \frac{\partial E_2}{\partial x} \langle E_3 E_4^* \rangle + E_1 E_2^* \frac{\partial E_3}{\partial x} \langle E_4^* \rangle + E_1 E_2^* E_3^* \frac{\partial E_4}{\partial x}. \quad (4.48)$$

But

$$\frac{\partial E_i}{\partial x} = \frac{i}{2k_0} \nabla_{Ti}^2 E_i,$$

therefore:

$$\frac{\partial}{\partial x} \langle E_1 E_2^* E_3 E_4^* \rangle = \frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T3}^2 + \nabla_{T3}^2 - \nabla_{T4}^2) \langle E_1 E_2^* E_3 E_4^* \rangle. \quad (4.49)$$

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The ‘scattering’ effect:

We have

$$(E_1 E_2^* E_3 E_4^*)(x+d) = (E_1 E_2^* E_3 E_4^*)(x) e^{\phi_1 - \phi_2 + \phi_3 - \phi_4} , \quad (4.50)$$

and

$$(E_1 E_2^* E_3 E_4^*)(x+d) = (E_1 E_2^* E_3 E_4^*)(x) + \frac{\partial(E_1 E_2^* E_3 E_4^*)}{\partial x} d . \quad (4.51)$$

Equating (4.50) and (4.51), and expanding the exponent, we get

$$(E_1 E_2^* E_3 E_4^*)(x+d) = (E_1 E_2^* E_3 E_4^*)(x) [1 + i(\phi_1 - \phi_2 + \phi_3 - \phi_4) - \frac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4)^2 + \dots] \quad (4.52)$$

So, truncating the expansion and taking the ensemble average, we have

$$\begin{aligned} \frac{\partial}{\partial x} \langle E_1 E_2^* E_3 E_4^* \rangle d &= -\frac{1}{2}(\phi_1 - \phi_2 + \phi_3 - \phi_4)^2 \langle E_1 E_2^* E_3 E_4^* \rangle \\ &= -\frac{1}{2} (4 \langle \phi^4 \rangle + 2 \langle \phi_1 \phi_3 \rangle + 2 \langle \phi_2 \phi_4 \rangle \\ &\quad - 2 \langle \phi_1 \phi_2 \rangle - 2 \langle \phi_1 \phi_4 \rangle - 2 \langle \phi_2 \phi_3 \rangle - 2 \langle \phi_3 \phi_4 \rangle) \langle E_1 E_2^* E_3 E_4^* \rangle , \end{aligned} \quad (4.53)$$

where we have used

$$\langle \phi_i \phi_j \rangle = \langle \phi_j \phi_i \rangle$$

and

$$\langle \phi_i \phi_i \rangle = \langle \phi_i^2 \rangle .$$

Now, proceeding as before, and remembering that (equation (4.43)),

$$\langle \phi_i \phi_j \rangle = k_0^2 \mu^2 \sigma(\eta_j, \zeta_j) d , \quad (4.54)$$

we can combine the ‘distance’ and ‘scattering’ effects to obtain the **fourth moment equation**

$$\begin{aligned} \frac{\partial m_4}{\partial x} &= -\frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T3}^2 + \nabla_{T3}^2 - \nabla_{T4}^2) m_4 \\ &\quad - k_0^2 \mu^2 \sigma(0,0) (2 + \sigma_{13} + \sigma_{24} - \sigma_{12} - \sigma_{14} - \sigma_{23} - \sigma_{34}) m_4 . \end{aligned} \quad (4.55)$$

This is more usefully often written in the slightly different form:

$$\begin{aligned} \frac{\partial m_4}{\partial x} &= -\frac{i}{2k_0} (\nabla_{T1}^2 - \nabla_{T3}^2 + \nabla_{T3}^2 - \nabla_{T4}^2) m_4 \\ &\quad - \beta (2 + f_{13} + f_{24} - f_{12} - f_{14} - f_{23} - f_{34}) m_4 , \end{aligned} \quad (4.56)$$

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where

$$\beta = k_0^2 \mu^2 \sigma(0, 0) \quad (4.57)$$

and

$$f_{i,j} = \sigma(y_i - y_j, z_i - z_j) / \sigma(0, 0) . \quad (4.58)$$

The quantity β defined above is the so-called *attenuation coefficient*, which is a useful parameter of physical significance. β is of course related to the attenuation of the mean field $\langle E \rangle$ by the medium, since

$$\langle E \rangle = E_0 e^{-\beta/2} ,$$

and to the 'unscattered' power $\langle E \rangle^2$ by

$$\langle E \rangle^2 = E_0 e^{-\beta} , \quad (4.59)$$

It has a further physical significance in terms of the mean free path of a photon in a random medium. Suppose the incident (electromagnetic) field is regarded as a unidirectional flux of photons. The incident field is attenuated exponentially like

$$e^{\mathbf{x}/x_m}$$

as it passes through the medium, where x_m is the mean free path of the photon in the medium. The number of photons in the unscattered flux is proportional to the unscattered power, so, comparing with (4.59), we see that β^{-1} may be interpreted as the mean free path of a photon in a random medium.

Let us now find the solutions for the second and fourth moment.

Solution of the second moment equation (*non-examinable*)

It is convenient to use the set of variables

$$\begin{aligned} \xi &= x_1 - x_2 , \quad \eta = y_1 - y_2 , \quad \zeta = z_1 - z_2 \\ X &= x_1 + x_2 , \quad Y = y_1 + y_2 , \quad Z = z_1 + z_2 \end{aligned} \quad (4.60)$$

and to set

$$\langle E_1 E_2^* \rangle e^{\beta x} = u(\beta x, \eta, \zeta) . \quad (4.61)$$

The equation for the second moment then can be written as

$$\frac{\partial u}{\partial(\beta x)} = -\frac{2i}{k_0 \beta} \left(\frac{\partial^2 u}{\partial Y \partial \eta} + \frac{\partial^2 u}{\partial Z \partial \zeta} \right) + \frac{\sigma(\eta, \zeta)}{\sigma(0, 0)} u . \quad (4.62)$$

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It is convenient to transform this equation using the transform pair

$$\begin{aligned} u(Y, \eta; Z, \zeta) &= \int \int \hat{u}(\eta, \zeta; \epsilon_1, \epsilon_2) e^{i(\epsilon_1 Y + \epsilon_2 Z)} d\epsilon_1 d\epsilon_2, \\ \hat{u}(\eta, \zeta; \epsilon_1, \epsilon_2) &= \frac{1}{2\pi} \int \int u(Y, \eta; Z, \zeta) e^{-i(\epsilon_1 Y + \epsilon_2 Z)} dY dZ. \end{aligned}$$

to obtain

$$\frac{\partial \hat{u}}{\partial(\beta x)} = B_1 \frac{\partial \hat{u}}{\partial \eta} + C_1 \frac{\partial \hat{u}}{\partial \zeta} + \frac{\sigma(\eta, \zeta)}{\sigma(0, 0)} \hat{u}, \quad (4.63)$$

where

$$B_1 = 2\epsilon_1/k\beta, \quad C_1 = 2\epsilon_2/k\beta.$$

The general solution of (4.63) is

$$\hat{u} = \hat{u}_0(\eta + B_1 \beta x, \zeta + C_1 \beta x) \exp \left[\int_0^{\beta x} \frac{\sigma(\eta + B_1(\beta x - \beta x'), \zeta + C_1(\beta x - \beta x'))}{\sigma(0, 0)} d(\beta x') \right], \quad (4.64)$$

where $\hat{u}_0$ is the solution of the transform equation (4.63) when $\sigma(\xi, \eta, \zeta) = 0$. The second moment then is given by the inverse transform, which, in our case where $Y = Z = 0$, reduces to

$$u(Y, \eta; Z, \zeta) = \int \int \hat{u} d\epsilon_1 d\epsilon_2. \quad (4.65)$$

If the incident field is a plane wave with amplitude E_0 at $x = 0$ and propagating parallel to the x -direction, then

$$\hat{u}_0 = \delta(\epsilon_1) \delta(\epsilon_2) \quad (4.66)$$

and from (4.64) and the inverse transform we have:

$$\begin{aligned} u &= E_0^2 \exp \left[\int_0^{\beta x} \frac{\sigma(\eta; \zeta) d(\beta x')}{\sigma(0, 0)} \right] \\ &= E_0^2 \exp \left[-\beta x \left(1 - \frac{\sigma(\eta; \zeta)}{\sigma(0, 0)} \right) \right]. \end{aligned} \quad (4.67)$$

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Solution of the fourth moment equation (*non-examinable*)

We shall now seek a solution of equation (4.56):

$$\begin{aligned} \frac{\partial m_4}{\partial x} = & -\frac{i}{2k_0}(\nabla_{T1}^2 - \nabla_{T3}^2 + \nabla_{T3}^2 - \nabla_{T4}^2)m_4 \\ & - k_0^2 \mu^2 \sigma_0 (2 + f_{13} + f_{24} - f_{12} - f_{14} - f_{23} - f_{34})m_4, \end{aligned} \quad (4.68)$$

We shall follow similar steps to those used to find a solution of the second moment equation, so we shall first make an appropriate change of variables, then use Fourier transforms.

Denote by L the scale size of the inhomogeneities transverse to the direction of propagation x , and define a new variable, scaling x by the so-called 'Fresnel length' kL^2 :

$$X = \frac{x}{kL^2} \quad (4.69)$$

Introduce also the following scaled variables:

$$\begin{aligned} \zeta_a &= (z_1 - z_2 - z_3 + z_4)/2L \\ \zeta_b &= (z_1 + z_2 - z_3 - z_4)/2L \\ \zeta_c &= (z_1 - z_2 + z_3 - z_4)/2L \\ Z &= (z_1 + z_2 + z_3 + z_4)/L \end{aligned} \quad (4.70)$$

and analogous ones relating in the y coordinate:

$$\begin{aligned} \eta_a &= (y_1 - y_2 - y_3 + y_4)/2L \\ \eta_b &= (y_1 + y_2 - y_3 - y_4)/2L \\ \eta_c &= (y_1 - y_2 + y_3 - y_4)/2L \\ Y &= (y_1 + y_2 + y_3 + y_4)/L \end{aligned} \quad (4.71)$$

We shall also define the parameter

$$\Gamma = k^3 \mu^2 \sigma_0 L^2 \quad (4.72)$$

For simplicity, we shall restrict the following to 2 dimensions, in the plane (x, z) . It will be straightforward to extend the final result to include the y coordinate. In this case then, and with the new variables defined above, the fourth moment equation becomes

$$\frac{\partial m_4}{\partial X} = -i \left(\frac{\partial^2 m_4}{\partial \zeta_a \partial \zeta_2} + \frac{\partial^2 m_4}{\partial \zeta_c \partial Z} \right) - 2\Gamma(1 - g(\zeta_a, \zeta_b, \zeta_c))m_4, \quad (4.73)$$

We shall now seek the solution in 2D, for a plane wave of unit amplitude normally incident onto the half-space $x > 0$. Equation (4.68) then simplifies

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further, since in this case the 4th moment is independent of the transverse direction Z , and all the field quantities in (4.73) can be written as functions of 2 new variables only. We have:

$$\frac{\partial m_4}{\partial X} = -i \frac{\partial^2 m_4}{\partial \zeta_a \partial \zeta_2} - 2\Gamma(1 - g(\zeta_a, \zeta_b))m_4 , \quad (4.74)$$

where now

$$g = f(\zeta_a) + f(\zeta_b) + -\frac{1}{2}f(\zeta_a + \zeta_b/2) - \frac{1}{2}f(\zeta_a - \zeta_b/2)$$

Similarly to the procedure followed to find the solution for the second moment, set

$$m_4 e^{2\Gamma X} = m ,$$

then use this in (4.74), and multiply the resulting equation by $e^{-2\Gamma X}$, to obtain

$$\frac{\partial m}{\partial X} = -i \frac{\partial^2 m}{\partial \zeta_a \partial \zeta_2} + 2\Gamma g m , \quad (4.75)$$

Again we transform this equation using Fourier transforms:

$$\begin{aligned} M &= \frac{1}{2\pi} \int \int m(\zeta_a, \zeta_b, X) e^{-i(\nu_a \zeta_a + \nu_b \zeta_b)} d\zeta_a d\zeta_b , \\ G &= \frac{1}{2\pi} \int \int g(\zeta_a, \zeta_b) e^{-i(\nu_a \zeta_a + \nu_b \zeta_b)} d\zeta_a d\zeta_b , \end{aligned}$$

and obtain

$$\frac{\partial M}{\partial X} = i\nu_a \nu_b M + 2\Gamma \int \int G(\nu_a - \nu'_a, \nu_b - \nu'_b) M(\nu'_a, \nu'_b, X) \quad (4.76)$$

In order to solve this integrodifferential equation, we shall now represent M as a series:

$$M = \sum_{n=0}^{\infty} M_n , \quad (4.77)$$

where

$$M_0 = M(\nu_a, \nu_b, 0) = \delta(\nu_a) \delta(\nu_b) , \quad (4.78)$$

which we take as the initial condition for (4.76). Using now the series (4.77) in (4.77) gives:

$$\frac{\partial M_n}{\partial X} = i\nu_a \nu_b M_n + 2\Gamma \int \int G(\nu_a - \nu'_a, \nu_b - \nu'_b) M_{n-1}(\nu'_a, \nu'_b, X) \quad (4.79)$$

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We can solve (4.79) starting from M_1 :

$$\frac{\partial M_n}{\partial X} = i\nu_a \nu_b M_n + 2\Gamma G(\nu_a, \nu_b) , \quad (4.80)$$

with initial condition $M_1(\nu_a, \nu_b, 0) = 0$. This has solution

$$M_1(\nu_a, \nu_b, X) = 2\Gamma \int_0^X G(\nu_a, \nu_b) e^{i\nu_a \nu_b (X-X_1)} dX_1 \quad (4.81)$$

Now we can use M_1 to solve for M_2 :

$$\frac{\partial M_2}{\partial X} = i\nu_a \nu_b M_2 + 2\Gamma \int \int G(\nu_a - \nu'_a, \nu_b - \nu'_b) M_1(\nu'_a, \nu'_b, X) d\nu'_a d\nu'_b \quad (4.82)$$

with initial condition $M_2(\nu_a, \nu_b, 0) = 0$. This has solution

$$\begin{aligned} M_2(\nu_a, \nu_b, X) &= \\ (2\Gamma)^2 &\int_0^X \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\nu_a - \nu_{a_1}, \nu_b - \nu_{b_1}) M_1(\nu_{a_1}, \nu_{b_1}, X_2) e^{i\nu_a \nu_b (X-X_2)} dX_2 d\nu_{a_1} d\nu_{b_1} \\ &= (2\Gamma)^2 \int_0^X \int_0^{X_2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(\nu_a - \nu_{a_1}, \nu_b - \nu_{b_1}) G(\nu_{a_1} - \nu_{a_2}, \nu_{b_1} - \nu_{b_2}, X_2) \times \\ &\quad e^{i\nu_a \nu_b (X-X_2) + i\nu_{a_1} \nu_{b_1} (X_2-X_1)} dX_2 d\nu_{a_1} d\nu_{b_1} \end{aligned}$$

and the n th term in the series is:

$$\begin{aligned} M_n(\nu_a, \nu_b, X) &= (2\Gamma)^n \int_0^X \int_0^{X_n} \dots \int_0^{X_2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} G(\nu_{a_1}, \nu_{b_1}) \\ &\quad \times G(\nu_{a_2} - \nu_{a_1}, \nu_{b_2} - \nu_{b_1}) \\ &\quad \times G(\nu_{a_3} - \nu_{a_2}, \nu_{b_3} - \nu_{b_2}) \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \cdot \\ &\quad \times G(\nu_a - \nu_{a_{n-1}}, \nu_b - \nu_{b_{n-1}}) \\ &\quad \times e^{i(\nu_{a_1} \nu_{b_1} (X_2-X_1) + \nu_{a_2} \nu_{b_2} (X_3-X_2) + \dots + \nu_a \nu_b (X-X_n))} \\ &\quad \times d\nu_{a_1} \dots d\nu_{a_{n-1}} d\nu_{b_1} \dots d\nu_{b_{n-1}} dX_1 \dots dX_n \end{aligned}$$

Replacing all the terms G by their inverse transform and carrying out all the possible integrals gives:

$$\begin{aligned} M_n(\nu_a, \nu_b, X) &= \frac{(2\Gamma)^n}{(2\pi)^{2n+2}} \int_0^X \int_0^{X_n} \dots \int_0^{X_2} \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} \prod_{j=1}^n g(\zeta_{a_j}, \zeta_b + Q_j) \\ &\quad \times e^{i\zeta_{a_j}(\nu_{a_j} - \nu_{a_{j-1}}) - i\nu_b \zeta_b} \\ &\quad \times d\zeta_b d\zeta_{a_1} \dots d\zeta_{a_n} \\ &\quad \times d\nu_{a_1} \dots d\nu_{a_{n-1}} dX_1 \dots dX_n , \end{aligned}$$

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where

$$Q_j = (\nu_a(X - X_n) + \nu_{a_{n-1}}(X_n - X_n) + \dots + \nu_{a_j}(X_{j+1} - X_j)) . \quad (4.83)$$

It is useful to interpret the subscripts n as the number of times that the field is scattered, contributing to a given term in the series, M_n . The “single scatter” approximation then corresponds to taking just the first term in the series, which is

$$M_1(\nu_a, \nu_b, X) = 2\Gamma \int_0^X G((\nu_a, \nu_b)) e^{i(\nu_a \nu_b (X - X_1))} dX_1 \quad (4.84)$$

and we recover the Born approximation.

In order to obtain higher order terms in the sum, further approximations are necessary, which of course involve further errors, beyond those incurred in truncating the series.

Considerations on the numerical solution of the moment equations

Because of the necessity for several approximations at each step, and because of the quite complicated analytical form of the higher order corrections, which do not readily yield physical insight, calculations for the fourth moment are most usefully carried out using *numerical approximations*.

We can write the equation for the fourth moment (equation (4.74)) in terms of operators as:

$$\frac{\partial m_4}{\partial X} = (A(X) + B(\Gamma, X)) m_4 , \quad (4.85)$$

where

$$A = i \frac{\partial^2}{\partial \zeta_a \partial \zeta_b} ; \quad B = -2\Gamma(1 - g(\zeta_a, \zeta_b))m . \quad (4.86)$$

We can write $C = A + B$. Difficulties can arise in the numerical solution for C , particularly for large values of Γ , when semi-discretization leads to a stiff system of differential equations. Furthermore, since there are two transverse variables, the matrices which operate on this system are of order N^4 , where N is the number of points in the discretization along each axis.

The formal solution of equation (4.85) over the range $(X, X + \Delta X)$ is

$$m_4(\zeta_a, \zeta_b, X + \Delta X) = e^{\int_X^{X+\Delta X} C(\Gamma, X') dX'} m_4(\zeta_a, \zeta_b, X) . \quad (4.87)$$

In the case of a plane wave, C does not vary with X , so

$$\int C dX = \Delta X C$$

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and the exact formal solution is

$$\begin{aligned} m_4(\zeta_a, \zeta_b, X + \Delta X) &= e^{\Delta X(A+B)} m_4(\zeta_a, \zeta_b, X) \\ &= \left(1 + \Delta X(A+B) + \frac{(\Delta X)^2}{2}(A+B)^2 + \dots \right) m_4(\zeta_a, \zeta_b, X). \end{aligned} \quad (4.88)$$

We first approximate this with the 'operator splitting' solution given by

$$\begin{aligned} m_4(\zeta_a, \zeta_b, X + \Delta X) &= e^{\Delta X A} e^{\Delta X B} m_4(\zeta_a, \zeta_b, X) \\ &= \left(1 + \Delta X A + \frac{(\Delta X)^2}{2} A^2 + \dots \right) \left(1 + \Delta X B + \frac{(\Delta X)^2}{2} B^2 + \dots \right) m_4(\zeta_a, \zeta_b, X). \end{aligned} \quad (4.89)$$

This is exact only if the operators A and B commute. We can see, by expanding the terms in (4.88) and comparing with (4.89), that the step-wise error in (4.89) is $O[(\Delta X)^3]$. However, the overall accuracy depends on the degree of commutativity between A and B in the strong operator topology or, in other words, on the quantity $\| (AB - BA)m_4 \|$. This quantity is indeed very small, so the method is very accurate.

This operator splitting can be applied when the irregularities in the medium have any given autocorrelation function with an outer scale, even if it is range-dependent.

The method is unconditionally stable and convergent, and can be applied even when there is strong scattering.

The method allows comparison of analytical and numerical intensity fluctuation spectra over a wide range of Γ and X .

Scattering effect of extended irregularities

It is interesting, and useful for acquiring some insight, to investigate heuristically the effect of the horizontal length scale. For a given form of the medium W and its statistics, what is the effect of changes in the length scale H ?

For the following heuristic analysis we shall *ignore diffraction* and examine only the scattering term in the parabolic equation for the wavefield (4.24).

Consider therefore a vertical layer consisting of the region $[x, x+d]$. Subdivide this into n thin subregions each of width $\Delta x = d/n$.

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Each of these subregions, for $j = 1, \dots, n$, imposes a normally-distributed phase change $\phi_j(z)$ with mean zero, whose variance is assumed to be given, say:

$$\langle \phi_j \rangle = 0, \quad \langle \phi_j^2(z) \rangle = \delta^2. \quad (4.90)$$

So since we are ignoring diffraction the wave emerging at $x + d$ has the form

$$E(x + d, z) = E(x, z) e^{i\phi(z)} \quad (4.91)$$

where

$$\phi(z) = \sum_{i=1}^n \phi_i(z).$$

Now, since ϕ is normally distributed, the mean of this phase modification is

$$\langle e^{i\phi} \rangle = e^{-\langle \phi^2 \rangle / 2} \quad (4.92)$$

so we want to examine the dependence of $\langle \phi^2 \rangle$ on H . Consider two extreme cases:

(1) H small, say $H \leq \Delta x$: Then we can treat ϕ_i, ϕ_j as independent for all $i \neq j$, so that

$$\begin{aligned} \langle \phi^2 \rangle &= \left\langle \left(\sum_{i=1}^n \phi_i(z) \right)^2 \right\rangle \\ &= \sum_{i=1}^n \langle \phi_i^2(z) \rangle \\ &= n\delta^2 \end{aligned} \quad (4.93)$$

so that scattering scales linearly with n

(2) H large, say $H \gg d$: Then we can suppose that the medium at each depth z is approximately constant over the interval $[x, x + d]$,

$$\phi_i(z) = \phi_j(z) \text{ for all } i, j,$$

so that

$$\langle \phi^2 \rangle = \langle [n\phi_1(z)]^2 \rangle = n^2 \delta^2. \quad (4.94)$$

Thus, increasing H magnifies the scattering effect of the medium.

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The scintillation index

The scintillation index S_I^2 is the normalised variance (sometimes referred to as ‘*mean square*’) of the intensity fluctuations:

$$S_I^2 = \frac{\langle I^2 \rangle - \langle I \rangle^2}{\langle I \rangle^2} , \quad (4.95)$$

where the intensity I is given by

$$I = EE^* .$$

S_I^2 is a measure of the fluctuations of the received signal by most devices, and is of course a fourth moment of the field.

5 The inverse scattering problem

We have so far only considered the *direct* scattering problem, i.e., given a wavefield u_i incident upon an inhomogeneity (this could be an interface such as an infinite surface or a finite, closed object, or an extended inhomogeneity such as a medium with varying refractive index), we have considered ways of finding the scattered field u_s , or equivalently the total field $u = u_i + u_s$.

The *inverse* scattering problem starts from the knowledge of the scattered field u_s , and asks questions about the inhomogeneities that produced it (for example their shape, or their refractive index) and about the source field.

This area of research is fairly new, because the nature of the problem gives rise to a mathematical problem which is *ill-posed*, and until about the '60's was not considered worth studying from a mathematical point of view.

Let's see what '*well-posed*' means. According to Hadamard, a problem is ***well-posed*** if

1. There exists a solution to the problem (**existence**)
2. There is at most one solution (**uniqueness**)
3. The solution depends continuously on the data (**stability**)

For a problem expressed as

$$Ax = y, \quad (5.1)$$

where A is an operator from a normed space X into a normed space Y , $A : X \mapsto Y$ the requirements listed above translate into the following properties of the operator A :

1. **A is surjective.** If it isn't, then equation (5.1) is not solvable for all $y \in Y$ (*non-existence*).

2. **A is injective.** If it isn't, then equation (5.1) may have more than one solution (*non-uniqueness*).

3. The **solution depends continuously on y** , i.e. $\forall$ sequences $x_n \in X$ with $Ax_n \rightarrow Ky$ as $n \rightarrow \infty$, it follows that $x_n \rightarrow x$ as $n \rightarrow \infty$. If this is not the case, then there may be cases when for $\|y' - y\| \ll 1$ we have $\|x' - x\| \gg 1$, small differences in y (e.g. small errors in the measurement or in the numerical computation give rise to large errors in the solution (*instability*)).

Absence of even one of these properties is likely to pose considerable difficulties in finding the solution to a problem.

5.1 Tikhonov regularisation

If all the above properties apply, then the inverse operator $A^{-1} : Y \mapsto X$ exists and is bounded, and

$$\|x\| \leq C \|y\|, \quad (5.2)$$

where $C = \|A^{-1}\|$.

If an inverse does exist for some y , but is not bounded, then there does not exist a constant C for which (5.2) holds for all $y \in A(X)$.

It is possible, though, even when A^{-1} is not bounded, but has dense range, to construct a family of bounded approximation to A^{-1} . A strategy for achieving this is the ***Tikhonov regularisation*** procedure, which provides a mean to cope with ill-posedness.

5.1 Tikhonov regularisation

Definition A regularisation strategy for $A : X \mapsto Y$ is a family of bounded linear operators $R_\alpha : Y \rightarrow X$ for $\alpha > 0$ such that

$$R_\alpha y \rightarrow A^{-1}y \text{ as } \alpha \rightarrow 0 \quad (5.3)$$

When it is not clear whether a solution to the inverse scattering problem for (5.1) exists, it is natural, as a first attempt at computing an approximate solution, to try to find an x to minimise $\|Ax - y\|$.

It is possible to demonstrate that (**Theorem**): *For every $y \in Y$, then $x' \in X$ satisfies*

$$\|Ax' - y\| \leq \|Ax - y\|.$$

if and only if x' solves the normal equation

$$A^*Ax' = A^*y, \quad (5.4)$$

where $A^* : Y \mapsto X$.

Equation (5.4) is still ill-posed, if the original scattering problem was ill-posed, but this ill-posedness can be removed by introducing a small perturbation, so replacing the original problem with the slightly perturbed one below:

$$\alpha x_\alpha + A^*Ax_\alpha = A^*y \quad (5.5)$$

for some small $\alpha > 0$.

5.1 Tikhonov regularisation

It is possible to prove that (**Theorem**):

If $\alpha > 0$, then the operator $(\alpha I + A^*A) : X \mapsto X$ has an inverse, which is bounded, with $\|(\alpha I + A^*A)^{-1}\| \leq \alpha^{-1}$.

Given a linear bounded operator $A : X \mapsto Y$, and $y \in Y$, the **Tikhonov functional** is defined by

$$J_\alpha = \|Ax - y\|^2 + \alpha \|x\|^2 \quad \forall x \in X \quad (5.6)$$

For $\alpha > 0$, the Tikhonov functional J_α , as defined above, has a unique minimum x_α given as the unique solution of the equation

$$\alpha x_\alpha + A^*Ax_\alpha = A^*y. \quad (5.7)$$

The solution of this equation can be written as $x_\alpha = R_\alpha y$, with

$$R_\alpha = (\alpha I + A^*A)^{-1}A^* : Y \mapsto X. \quad (5.8)$$

$x_\alpha = R_\alpha y$ is referred to as the *Tikhonov regularisation solution* of (5.1).

This strategy then approximates the actual solution $x = A^{-1}y$ by the regularised solution x_α , given y . In general, a y_δ will be known, which differs from y by some error δ (for example because it is experimental data):

$$\|y_\delta - y\| \leq \delta. \quad (5.9)$$

It is useful to be able to approximate the error involved in the regularisation, and to relate it to the error associated with incorrect initial data δ . Let's write

$$x_{\alpha(\delta)} - x = R_\alpha y_\delta - R_\alpha y + R_\alpha Ax - x. \quad (5.10)$$

Then, by the triangle inequality we have the estimate

$$\|x_{\alpha(\delta)} - x\| \leq \delta \|R_\alpha\| + \|R_\alpha Ax - x\| \quad (5.11)$$

This decomposition shows that the error consists of two parts: the first term reflects the influence of the incorrect data, and the second term is due to the approximation error between R_α and A^{-1} .

The regularisation scheme requires a strategy for choosing the parameter α on the basis of the error δ in the data, in order to achieve an acceptable total error for the regularised solution.

6 Methods for solving the inverse scattering problem

Inverse problems have a variety of very important practical applications, ranging from the detection of land mines, to medical imaging, analysis of sub-surface strata for oil and gas recovery, reconstruction and detection of craft, missiles and submarines, non-destructive testing of materials and structures, and many more.

As mentioned previously, there are several types of inverse scattering problems. We shall concentrate first on the problem of reconstructing the geometry of the scatterer, then we shall consider the problem of reconstructing the refractive index. We shall only present a few simple results.

The main book available on inverse scattering is "Inverse Acoustic and Electromagnetic Scattering", by Colton and Kress [4], which is very good, but far beyond the scope of this course. Several good and useful reviews are also available, for example: Sleeman BD (1982) IMA J. Appl. Math. **29** 113-142, Colton D (2003) Inside Out: Inverse Problems **47** 67-110, Potthast R (2006) Inverse Problems **22** R1-R47.

Inverse problems have been treated from many points of view.

- Some exact solutions, depending on the geometry of the scatterer, are available. They are usually based on expressing the surface of the scatterer parametrically in a coordinate system in which the Helmholtz equation is separable.
- Some methods exploit the properties of the far field in order to construct an analytical continuation of the far field into the near field of the scatterer, and the circle of minimum radius enclosing the scatterer, then determine enough points on the scatterer to approximate its shape sufficiently. The *method of Imbriale and Mittra* comes in this category.

These methods need some a priori knowledge of the boundary condition at the surface of the scatterer, so will not work for many real problems where only very limited of the scatterer is available.

They will usually not work in the 'resonance region', where the size of the incident wavelength is comparable to the size of the scatterer.

- Other methods which exploit the properties of the far field use an 'indicator function' to construct the shape of the scatterer on a grid covering some area which is 'searched' for the unknown scatterer. The *linear*

sampling method and some of its variants are in this category. They are usually based on a factorization of the 'far field operator', which maps the incident field onto the far field.

They do not need a priori knowledge of the scatterer

They can be used in the resonance region, and can also be used for scatterer which are not simply connected.

In principle, though, these methods need knowledge of the far field in *all* direction, which of course is very often impossible to obtain.

- A number of methods based on *iterative* procedures are also available. They need some a priory knowledge to make an initial 'guess'. In practice, this often means linearising the problem.

They usually need to solve the forward problem at least once at each iteration step

When the data are sufficiently good, and the scatterer or inhomogeneities sufficiently 'smooth', they can provide very high-quality reconstruction.

These methods are particularly suited to lower frequencies, and for problems of scattering by extended inhomogeneities.

They will not work, in general, for many practical problems where only limited knowledge of the scatterer is available, also not for high frequencies, nor in the 'resonance region' where the size of the incident wavelength is comparable to the size of the scatterer or of the inhomogeneities.

The first two methods we are going to present here are based on the properties of the far field. Therefore, before we proceed, we shall first consider some important results concerning the far field amplitude. Recall that, in the direct scattering problem, given an incident (time-harmonic) field ϕ_i with direction defined by the incident wavevector $\mathbf{k}$ on a (bounded) scatterer with boundary ∂V , we seek the total field

$$\phi = \phi_i + \phi_s ,$$

such that ϕ_s obeys the Helmholtz equation with suitable boundary conditions

on ∂V , and the radiation condition at infinity, which can be expressed as

$$r^{1/2} \left(\frac{\partial \phi_s}{\partial r} - ik\phi_s \right) \rightarrow 0 \quad \text{as } r \rightarrow \infty \quad (\text{in 2D}) \quad (6.1)$$

$$r \left(\frac{\partial \phi_s}{\partial r} - ik\phi_s \right) \rightarrow 0 \quad \text{as } r \rightarrow \infty \quad (\text{in 3D}), \quad (6.2)$$

with $k = |\mathbf{k}|$. Then there exists a function $f(\hat{\mathbf{x}}, \mathbf{k})$ such that

$$\phi_s(\hat{\mathbf{x}}, \mathbf{k}) = \frac{e^{kr}}{r^{1/2}} \left(f(\hat{\mathbf{x}}, \mathbf{k}) + O\left(\frac{1}{r^{1/2}}\right) \right) \quad (\text{in 2D}) \quad (6.3)$$

$$\phi_s(\hat{\mathbf{x}}, \mathbf{k}) = \frac{e^{kr}}{r} \left(f(\hat{\mathbf{x}}, \mathbf{k}) + O\left(\frac{1}{r}\right) \right) \quad (\text{in 3D}) \quad (6.4)$$

as $r \rightarrow \infty$. Here $\hat{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|$, and $r = |\mathbf{x}|$.

The function f is called the **far field amplitude**, or also **directivity pattern**. It is sometimes useful to expand $f(\hat{\mathbf{x}}, \mathbf{k})$ in terms of spherical harmonics:

$$f(\hat{\mathbf{x}}, \mathbf{k}) = \sum_{n=0}^{\infty} \sum_{m=-n}^n A_{nm} Y_n^m(\hat{\mathbf{x}}) \quad (6.5)$$

It is well known and it can be proven (see, e.g., Colton and Kress 1992), that

Theorem: *exact knowledge of the far field amplitude f uniquely determines the scatterer V .*

Note that this tells us that we can use f to determine the scatterer, but it doesn't tell us *how!* nor does it tell us whether we can still reconstruct the scatterer without *exact* knowledge of f , which is always the case in real life, where the far field is known only at a finite, discrete number of points, and with some experimental error and noise.

Another theorem, due to Müller (1955), characterises the functions admissible as far field amplitudes.

Theorem: *A necessary and sufficient condition in n dimension for a function $f(\hat{\mathbf{x}}, \mathbf{k})$ defined on the unit sphere S^{n-1} to be a far field amplitude is that:*

there exists a harmonic function $H(\hat{\mathbf{x}}, \mathbf{k})$, analytic for all $\mathbf{x} \in \mathbb{R}^n$, and such that $H(\hat{\mathbf{x}}, \mathbf{k}) = f(\hat{\mathbf{x}}, \mathbf{k})$ on S^{n-1} , and further has the property:

$$\int_{r=R} |H(\hat{\mathbf{x}}, \mathbf{k})|^2 ds = O(e^{2|\mathbf{k}|CR}) \quad , \quad (6.6)$$

where C is a non-negative constant.

When this condition is satisfied, there exists a **unique** function $\Phi(\hat{\mathbf{x}}, \mathbf{k})$ which satisfies the Sommerfeld radiation condition and is a regular solution of the Helmholtz equation for $|\mathbf{x}| > C$, such that

$$\Phi(\hat{\mathbf{x}}, \mathbf{k}) = \frac{e^{ikr}}{r^{1/2}} \left(f(\hat{\mathbf{x}}, \mathbf{k}) + O\left(\frac{1}{r^{1/2}}\right) \right) \quad (\text{in 2D}) \quad (6.7)$$

$$\Phi(\hat{\mathbf{x}}, \mathbf{k}) = \frac{e^{ikr}}{r} \left(f(\hat{\mathbf{x}}, \mathbf{k}) + O\left(\frac{1}{r}\right) \right) \quad (\text{in 3D}) \quad (6.8)$$

as $r \rightarrow \infty$.

The constant C in (6.6) gives the radius of the sphere outside which $\Phi(\hat{\mathbf{x}}, \mathbf{k})$ is defined. In other words, the sources generating the given far field are located within a sphere of radius C .

From the uniqueness of $\Phi(\hat{\mathbf{x}}, \mathbf{k})$ and $\phi_s(\hat{\mathbf{x}}, \mathbf{k})$, it follows that

$$\phi_s(\hat{\mathbf{x}}, \mathbf{k}) = \Phi(\hat{\mathbf{x}}, \mathbf{k}) \quad \text{for } |\mathbf{x}| > C. \quad (6.9)$$

Thus, an important problem to be considered is that of locating the region containing the sources that generate $\Phi(\mathbf{n}, \mathbf{k})$. Some methods, which we shall not consider here, seek to construct an analytic continuation of $\Phi(\mathbf{n}, \mathbf{k})$ into the region $|\mathbf{x}| \leq C$. In these methods, the total field is usually expanded in a series, then the geometry of a sound-soft scatterer is then determined by locating points at which the total field vanishes. They cannot treat scatterers with other boundary conditions, and they rely on the series expansion for the total field to be rapidly convergent. Procedures based on analytical continuations are inherently ill-posed and subject to numerical instability. Other methods, such as optimization and linear sampling, don't rely on a priori knowledge of the scatterer, can treat both Dirichlet and Neumann boundary conditions, and are more stable. They usually rely, though, on knowledge of the far field in all directions.

Before describing other methods, we shall define the **far field operator** and the **Herglotz function**.

Given a far field amplitude $f(\hat{\mathbf{y}}, k\hat{\mathbf{x}})$, the **Far Field Operator** F is defined by:

$$(Fg)(\hat{\mathbf{x}}) = \int_{S_1} f(\hat{\mathbf{y}}, k\hat{\mathbf{x}}) g(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}), \quad (6.10)$$

where S_1 is the unit sphere, and $g(\hat{\mathbf{y}})$ is a suitably well-behaved function on S_1 ($g \in L^2(S_1)$).

The **Herglotz wave function** with kernel g is defined by

$$v_g(\hat{\mathbf{x}}) = \int_{S_1} e^{-ik\hat{\mathbf{y}} \cdot \hat{\mathbf{x}}} g(\hat{\mathbf{y}}) ds(\hat{\mathbf{y}}). \quad (6.11)$$

It is effectively a superposition of plane waves. Herglotz wave functions satisfy the Helmholtz equation. Corresponding to an incident Herglotz wave function v_g^i in a scattering problem, we can associate a scattered field v_g^s and a far field amplitude v_g^∞ .

The Far Field Operator (6.10) can be seen to be the scattered field corresponding to an incident field defined by a Herglotz function with kernel g : $v_g(\hat{\mathbf{x}})$, given by (6.11) above.

The following theorem is important for deriving some of the inverse methods illustrated here. It relates the far field (rather surprisingly!) to the solutions of the Dirichlet problem inside the scatterer V (*interior Dirichlet problem*).

Theorem: *The Far Field Operator F for the exterior Dirichlet problem is injective with dense range*

if and only if

*$\nexists$ a non-trivial Herglotz wavefunction that vanishes on ∂V
(i.e. which is a solution of the interior Dirichlet problem)*

This means that for the far field patterns to be complete in $L^2(S_1)$, the eigenvalue k^2 must not coincide with the eigenvalues of the interior Dirichlet problem.

6.1 Optimization method

6.1 Optimization method

A method which was described by Colton & Monk (1987), for the scattering from an acoustically soft surface S , recasts the inverse scattering problem in terms of an optimization.

Suppose that a time-harmonic plane wave

$$\phi_i = e^{ik\mathbf{w} \cdot \mathbf{x}} \quad (6.12)$$

is incident upon a scattering surface S that encloses the origin. The total potential $\phi = \phi_i + \phi_s$ satisfies the Helmholtz equation and the boundary condition

$$\phi(\mathbf{x}) = 0, \quad \text{for } \mathbf{x} \text{ on } S. \quad (6.13)$$

We make the following assumptions:

1. k^2 is *not* one of the interior eigenvalues of the interior Dirichlet problem (this is necessary for existence and uniqueness of the solution, but in practice the errors in a numerical calculations will be sufficient to ensure that the eigenvalues of the interior Dirichlet problem are avoided);
2. the scatterer is "starlike", i.e. its surface S can be represented in the form

$$\mathbf{x} = r_s(\mathbf{e})\mathbf{e},$$

where $\mathbf{e} = \mathbf{x}/(|\mathbf{x}|)$, and r_s is single-valued (i.e. the surface of the scatterer can be parametrised by a single-valued function of angle - this is not strictly necessary, but is convenient for the calculations).

Given the far field amplitude $f(\mathbf{w}, \mathbf{e}, k)$ (in 3D) defined by

$$\phi_s \sim \left(\frac{1}{kr} \right) e^{ikr} f(\mathbf{w}, \mathbf{e}, k) \quad \text{as } kr \rightarrow \infty, \quad (6.14)$$

at fixed k , the problem is to determine the function $r_s(\mathbf{e})$ that specifies the scattering surface S .

Colton & Monk relate the far field $f(\mathbf{w}, \mathbf{e}, k)$ to a function $\psi(\mathbf{x}, k)$ that correspond to the scattered potential inside S induced by a point at the origin, i.e the function $\psi(\mathbf{x}, k)$ which satisfies

$$(\nabla^2 + k^2)\psi(\mathbf{x}, k) = 0 \quad \mathbf{x} \text{ inside } S, r_s = |\mathbf{x}|, \quad (6.15)$$

with

$$\psi(\mathbf{x}, k) = \frac{e^{ikr_s}}{4\pi r_s}, \quad \mathbf{x} \text{ on } S, \quad (6.16)$$

6.1 Optimization method

By using Green's function formalism applied to the potential $\phi(\mathbf{x})$, with the Green's function

$$G(\mathbf{x}, \mathbf{y}) = \frac{e^{ikr}}{4\pi r}, \quad \text{with } r = |\mathbf{y} - \mathbf{x}|,$$

in the region outside S , we can write

$$\phi(\mathbf{x}) = \phi_i(\mathbf{x}) - \frac{1}{4\pi} \int_S \frac{e^{ikr}}{r} \frac{\partial \phi(\mathbf{y})}{\partial n} ds(\mathbf{y}), \quad (6.17)$$

where n denotes the outward normal from s . It follows that the far field amplitude f has the representation

$$f(\mathbf{w}, \mathbf{e}, k) = -\frac{k}{4\pi} \int_S e^{-ik\mathbf{e}\cdot\mathbf{y}} \frac{\partial \phi(\mathbf{y})}{\partial n} ds(\mathbf{y}). \quad (6.18)$$

Now define S_1 to be the sphere of unit radius and centre at the origin. The above identity (6.18) can be multiplied by a suitable function $g(\mathbf{e})$ and integrated with respect to $\mathbf{e}$ over the unit sphere S_1 , to get

$$-\frac{4\pi}{k} \int_{S_1} f(\mathbf{w}, \mathbf{e}, k) g(\mathbf{e}) d\mathbf{e} = \int_S \psi(\mathbf{y}) \frac{\partial \phi(\mathbf{y})}{\partial n} ds(\mathbf{y}). \quad (6.19)$$

where

$$\psi(\mathbf{y}) = \int_{S_1} g(\mathbf{e}) e^{-ik\mathbf{e}\cdot\mathbf{y}} d\mathbf{e} \quad (6.20)$$

is a Herglotz wave function and satisfies the Helmholtz equation if the kernel function $g(\mathbf{e})$ is sufficiently smooth.

It is now assumed that the domain inside S is such that the interior potential $\psi(\mathbf{x})$ defined by (6.16) and (6.18) can be represented as an Herglotz function. In this case, the integral (6.19) has value unity. This follows since, from (6.19) and (6.16), and using the boundary condition, we have:

$$\begin{aligned} -\frac{4\pi}{k} \int_{S_1} f(\mathbf{w}, \mathbf{e}, k) g(\mathbf{e}) d\mathbf{e} &= \frac{1}{4\pi} \int_S \frac{e^{ikr}}{r} \frac{\partial \phi(\mathbf{y})}{\partial n} ds(\mathbf{y}) \\ &= \frac{1}{4\pi} \int_S \left[\frac{\partial \phi}{\partial n} \frac{e^{ikr}}{r} - \phi \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] ds(\mathbf{y}) \\ &= \frac{1}{4\pi} \int_S \left[\frac{\partial \phi_i}{\partial n} \frac{e^{ikr}}{r} - \phi_i \frac{\partial}{\partial n} \left(\frac{e^{ikr}}{r} \right) \right] ds(\mathbf{y}) \end{aligned} \quad (6.21)$$

The last step follows from the fact that the integral

$$I = \int_{S'} \left[G \frac{\partial \phi_s}{\partial n} \right] ds(\mathbf{y}), \quad (6.22)$$

6.1 Optimization method

is invariant with respect to any surface S' on or outside S , by virtue of Green's formula applied to ϕ_s and G . Taking S' to be a sphere of large radius R_0 , one finds, by using the radiation condition satisfied by ϕ_s , that $I \rightarrow 0$ as $R_0 \rightarrow \infty$, hence $I \equiv 0$. Finally, the integral (6.21) is seen to have the value $\phi_i(0) = 1$, from Green's formula applied to ϕ_i and G , with $(\nabla^2 + k^2)G = \delta(\mathbf{x})$. Thus

$$-\frac{4\pi}{k} \int_{S_1} f(\mathbf{w}, \mathbf{e}, k) g(\mathbf{e}) ds(\mathbf{e}) = 1 \quad (6.23)$$

for all directions of incidence $\mathbf{w}$.

The problem is now specified by the two identities (6.23) and (6.16), which can be used to determine first $g(\mathbf{e})$, then r_s .

Colton & Monk (1987) accordingly formulate the optimization problem to minimise

$$\sum_{n=1}^N \left| \int_{S_1} \frac{4\pi}{k} f(\mathbf{w}_n, \mathbf{e}, k) g(\mathbf{e}) ds(\mathbf{e}) + 1 \right|^2 \quad (6.24)$$

with respect to $g(\mathbf{e})$ from a suitable function class, for a finite number N of incident directions. Given g , hence ψ from equation (6.20), the second identity (6.16) leads to the optimization problem of minimizing

$$\int_{S_1} \left| \psi(r_s(\mathbf{e})) - \frac{e^{ikr_s}}{4\pi r_s} \right|^2 ds(\mathbf{e}) \quad (6.25)$$

with respect to $r_s(\mathbf{e})$ from a suitable function class.

The estimate for r_s gives an approximation to the surface S .

Colton & Monk (1987) give results for several axially symmetric problems, using trial functions in the form of Fourier series in the azimuthal angle. Their results give excellent reconstructions for a variety of shapes, such as the oblate spheroid, the "peanut" shape, and the "acorn" shape.

6.2 The linear sampling method

6.2 The linear sampling method

This method, first proposed by Colton and Kirsch (1996), is also based on the properties of the far field pattern used in the previous section. The basic idea behind it, as its name suggests, is to choose points z in a range known to include the scatterer and provide a scheme for deciding whether the point is on the scatterer. It allows scatterers which are not simply connected, either penetrable or impenetrable, and - when impenetrable - does not require knowledge of the boundary conditions. It was originally started by numerical observations about the behaviour of the kernel of the integral equation for the far field of a point source, and has been successfully applied in a variety of cases. Its mathematical basis hasn't yet been rigorously proven, but we can see how it can be heuristically justified with relatively simple observations.

The linear sampling method therefore also makes use of the Far Field Operator defined earlier:

$$(Fg)(\hat{\mathbf{x}}) = \int_{S_1} f(\hat{\mathbf{y}}, k\hat{\mathbf{x}})g(\hat{\mathbf{y}})ds(\hat{\mathbf{y}}) , \quad (6.26)$$

where $f(\hat{\mathbf{y}}, k\hat{\mathbf{x}})$ is the measured (known) far field, $g(\hat{\mathbf{y}})$ is a suitably well-behaved function on S_1 , but S_1 is now the unit sphere centred on an arbitrary point z .

It is then concerned with solving the integral equation

$$(Fg)(\hat{\mathbf{x}}) = G_\infty(\hat{\mathbf{x}}, \mathbf{z}) . \quad (6.27)$$

where F is the Far Field Operator, and $G_\infty(\hat{\mathbf{y}}, \mathbf{z})$ is the far field pattern

$$G_\infty(\hat{\mathbf{x}}, \mathbf{z}) = \frac{1}{4\pi} e^{-ik\hat{\mathbf{x}} \cdot \mathbf{z}} \quad (6.28)$$

of a point source centred at $\mathbf{z}$:

$$G(\mathbf{x}, \mathbf{z}) = \frac{1}{4\pi} \frac{e^{-ik|\mathbf{x}-\mathbf{z}|}}{|\mathbf{x}-\mathbf{z}|} , \quad \mathbf{x} \neq \mathbf{z} , \quad (6.29)$$

and we assume $\mathbf{z}$ is on the scatterer D .

Under the assumption that k^2 is not a Dirichlet eigenvalue of the negative Laplacian in D (i.e. not an eigenvalue of the internal Dirichlet problem, as before), it can be shown that

Theorem: if $\mathbf{z} \in D$, $\forall \epsilon > 0 \exists$ a solution $g(\cdot, \mathbf{z})$ of the inequality

$$\|Fg - G_\infty\| < \epsilon \quad (6.30)$$

6.2 The linear sampling method

such that

$$\|g(\cdot, \mathbf{z})\| \rightarrow \infty \text{ and } \|v_g(\cdot, \mathbf{z})\| \rightarrow \infty \text{ as } \mathbf{z} \rightarrow \partial D, \quad (6.31)$$

where $v_g(\cdot, \mathbf{z})$ is the Herglotz function with kernel $g(\cdot, \mathbf{z})$.

This theorem effectively states that, whatever the incident field on a scatterer D , the far field is arbitrarily close to the far field of a point source centred in D , thus providing a way of calculating the Herglotz kernel g that relates the far field to the incident field. It further provides a way of identifying the surface of the scatterer D as the region where $\|g(\cdot, \mathbf{z})\|$ becomes unbounded.

The linear sampling method consists in the following steps:

First, find a g that minimises

$$\|Fg - G_\infty\|$$

This problem is ill-posed, and a solution therefore is found by regularising and minimising

$$\|Fg - G_\infty\|^2 + \alpha \|g\|^2 \quad (6.32)$$

The solution is sought for each $\mathbf{z}$ on a grid covering some area where we search for the unknown scatterer, hence the name ‘sampling method’.

Secondly, the unknown boundary of the scatterer ∂D is then found by looking for the points $\mathbf{z}$ where $\|g(\cdot, \mathbf{z})\|$ begins to increase sharply.

There are mathematical difficulties with this scheme, because in fact a solution to the minimisation problem is not guaranteed to exist if $\mathbf{z}$ is not on ∂D . To avoid these problems, a modified scheme has been proposed, which factorises the operator F using self-adjoint operators defined on ∂D (*‘factorisation method’*, Kirsch 1998). Other modified schemes are also in use.

6.3 Inverse scattering in the Born approximation

6.3 Inverse scattering in the Born approximation

When the scattering is sufficiently weak, the inverse scattering problem can be linearised and solved using the first Born (or Rytov) approximation (see Chapter 3, where these approximations are introduced for the direct scattering problem). In this case, the (known) scattered field is written as the first Born (or Rytov) solution of the direct scattering problem, then the Fourier transform of the scattered field is related to the Fourier transform of the ‘scattering potential’ of the object, or medium, thus formally solving the inverse problem.

We shall consider first the Born approximation. We recall (see Chapter 3) that, given some inhomogeneity with refractive index $n(\mathbf{r})$, and a non-scattering background with refractive index 1, the total field satisfies

$$\nabla^2 \psi + k^2(\mathbf{r})\psi = 0, \quad (6.33)$$

where

$$k(\mathbf{r}) = k_0 n(\mathbf{r}) = k_0(1 + n_\delta(\mathbf{r})), \quad (6.34)$$

We shall assume $n_\delta(\mathbf{r}) \ll 1$. Substituting $k_0 n(\mathbf{r})$ into (6.33) we get:

$$\nabla^2 \psi + k_0^2(\mathbf{r})\psi = -k_0^2(n^2(\mathbf{r}) - 1)\psi \equiv -V(\mathbf{r})\psi, \quad (6.35)$$

and the scattered field is then given by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi(\mathbf{r}')] d\mathbf{r}'. \quad (6.36)$$

The total field is then given by

$$\psi = \psi_i(\mathbf{r}) + \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi(\mathbf{r}')] d\mathbf{r}', \quad (6.37)$$

and the scattered field can be approximated first Born approximation by

$$\psi_s(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}')\psi_i(\mathbf{r}')] d\mathbf{r}', \quad (6.38)$$

Here $G(\mathbf{r} - \mathbf{r}')$ is the free space Green’s function in 3 dimension, i.e.

$$G(\mathbf{r} - \mathbf{r}') = \frac{e^{ik_0 r}}{r}. \quad (6.39)$$

We shall now use the following representation for

$$\frac{e^{ik_0 r}}{r} :$$

6.3 Inverse scattering in the Born approximation

(see, e.g. Banos 1966, Wolf 1969):

$$G(\mathbf{r} - \mathbf{r}') = \frac{ik_0}{2\pi} \int \int_{-\infty}^{\infty} \frac{1}{m} e^{ik_0[p(x-x') + q(y-y') + m(z-z')]} dp dq , \quad (6.40)$$

where:

$$m = (1 - p^2 - q^2)^{1/2} \text{ when } (p^2 + q^2) < 1 \quad (6.41)$$

$$m = i(p^2 + q^2 - 1)^{1/2} \text{ when } (p^2 + q^2) > 1 \quad (6.42)$$

If we now substitute this expression for the Green's function into equation (6.38), we obtain:

$$\psi_s(\mathbf{r}) = \int \int_{-\infty}^{\infty} A^{(\pm)}(p, q; p_0, q_0) e^{ik_0(px + qy \pm mz)} dp dq , \quad (6.43)$$

where

$$A^{(\pm)}(p, q; p_0, q_0) = -\frac{ik_0}{8\pi^2 m} \int V \mathbf{r}' e^{ik_0[(p-p_0)x' + (q-q_0)y' \pm (m-m_0)z']} d\mathbf{r}' . \quad (6.44)$$

Here, the upper sign(+) applies in the region $\mathcal{R}^+$ where $z - z' > 0$, and the lower one (-) in the region $\mathcal{R}^-$ where $z - z' < 0$. Equation (6.43) represents the scattered field as an angular spectrum of plane waves, and the spectral amplitude function $A^{(\pm)}(p, q; p_0, q_0)$ is expressed in term of the scattering potential by (6.44). For homogeneous waves, i.e. when m is real, we obtain the relation:

$$A^{(\pm)}(p, q; p_0, q_0) = -\frac{ik_0}{8\pi^2 m} \hat{F}[k_0[(p - p_0), k_0(q - q_0), k_0 \pm (m - m_0)]] , \quad (6.45)$$

where $\hat{F}$ is the Fourier inverse of F :

$$\hat{F}(u, v, w) = \frac{1}{(2\pi)^3} \int F(x, y, z) e^{ik_0(ux + vy + wz)} dx dy dz . \quad (6.46)$$

Consider now the scattered field ψ_s in two fixed planes $z = z^+$ and $z = z^-$, situated respectively in $\mathcal{R}^+$ and $\mathcal{R}^-$.

Now, by taking the inverse Fourier transform of (6.43), with z at the fixed values z^+ and z^- , we obtain

$$A^{(\pm)}(p, q; p_0, q_0) = k_0^2 e^{\mp ik_0 m z^\pm} \hat{\psi}_s(k_0 p, k_0 q, z^\pm) , \quad (6.47)$$

where

$$\hat{\psi}_s(u, v, z^\pm) = \frac{1}{(2\pi)^2} \int \int_{-\infty}^{\infty} e^{-i(ux + vy)} dx dy \quad (6.48)$$

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is the inverse Fourier transform of ψ_s with respect to the variables x and y . Now, comparing (6.47) and (6.47), and using $m = (1 - p^2 - q^2)^{1/2}$, we obtain

$$\hat{V}(u', v', w'^{\pm}) = \frac{iw}{\pi} e^{\mp i w z^{\pm}} \hat{\psi}_s(u, v, w^{\pm}) , \quad (6.49)$$

where

$$\begin{aligned} u' &= u - k_0 p_0 \\ v' &= v - k_0 q_0 \\ w' &= \pm w - k_0 m_0 \end{aligned} \quad (6.50)$$

and

$$w = (k_0^2 - u^2 - v^2)^{1/2} . \quad (6.51)$$

Equation (6.49) shows that some of the three-dimensional Fourier components of the scattering potential v , and therefore the unknown refractive index, can be immediately determined by the two-dimensional components of the scattered field in the two planes $z = z^+$ and $z = z^-$.

Note that (6.49) is valid only for those two-dimensional Fourier components of $\hat{\psi}_s$ and ψ_s about which the information is carried by homogeneous waves, i.e. those for which $u^2 + v^2 \leq k_0^2$. In general, it is impossible to reconstruct inverse data associated with the high spectral components for which the information is carried by evanescent waves, because these waves decay very rapidly from the scatterer and do not contribute to the far field. This limitation arises because the problem is ill-posed.

We saw earlier that one way to obviate the limitations caused by ill-posedness is to use the Tikhonov (or other) regularization. In this case then, if we represent by $\mathbf{A}$ the integral operator in (6.38):

$$\mathbf{A}V(\mathbf{r}) = \int G(\mathbf{r} - \mathbf{r}') V(\mathbf{r}') \psi_i(\mathbf{r}') d\mathbf{r}' , \quad (6.52)$$

then the problem we need to solve is

$$\mathbf{d} = \mathbf{A}V(\mathbf{r}) \quad (6.53)$$

where $\mathbf{d}$ is the vector of the scattered field measurements. This can be regularised by minimising the Tikhonov functional

$$J_{\alpha} = \| \mathbf{A}V(\mathbf{r}) - \mathbf{d} \|^2 + \alpha \| x \|^2 \quad (6.54)$$

with the penalty parameter α usually chosen based on knowledge of the noise level.

6.3 Inverse scattering in the Born approximation

Using the first Born approximation for the inverse scattering problem reduces the non-linear inverse problem to a completely linear one.

We can retain some non-linearity either by adding higher order terms in the Born approximation, or by using the ***distorted-wave Born approximation*** (DWBA). In the DWBA, instead of approximating the ‘zero order’ solution with the incident field as in the first Born illustrated above, we start with a perturbed field, in other words, instead of writing the refraction index as

$$n(\mathbf{r}) = 1 + n_\delta , \quad (6.55)$$

we write

$$n^2(\mathbf{r}) = n_0^2(\mathbf{r}) + \epsilon n_1 + \epsilon^2 n_2 + \dots \quad (6.56)$$

The DWBA then is obtained by seeking a solution of the Helmholtz equation (6.33) in the form:

$$\psi(\mathbf{r}) = \psi_0(\mathbf{r}) + \epsilon \psi_1(\mathbf{r}) + \dots \quad (6.57)$$

The solution terms in this series can be computed by solving:

$$\begin{aligned} (\nabla^2 + k_0^2 n_0^2 \mathbf{r}) \psi_0 &= 0 \\ (\nabla^2 + k_0^2 n_0^2 \mathbf{r}) \psi_1 &= -k_0^2 n_1 \psi_0 \\ (\nabla^2 + k_0^2 n_0^2 \mathbf{r}) \psi_2 &= -k_0^2 n_2 \psi_0 - k_0^2 n_1 \psi_1 \\ &\dots \end{aligned} \quad (6.58)$$

So the integral equation corresponding to (6.38) is now:

$$\psi_s(\mathbf{r}) = \int G^{(k)}(\mathbf{r} - \mathbf{r}') [V(\mathbf{r}') \psi_i(\mathbf{r}')] d\mathbf{r}' , \quad (6.59)$$

and $G^{(k)}(\mathbf{r} - \mathbf{r}')$ is not the free space Green’s function any more. If $n_0^2(\mathbf{r}) = 1$, then the DWBA coincides with the Born approximation. In the DWBA it is also possible of course to go to higher terms and include more iterations.

It should be noted, though, that in general, if the measured data is contaminated with noise, so that the actual total field ψ^a is:

$$\psi^a(\mathbf{r}) = \psi(\mathbf{r}) + \Delta(\mathbf{r}) , \quad (6.60)$$

where $\Delta(\mathbf{r})$ is the noise, then

$$\psi_s(\mathbf{r}) = \psi(\mathbf{r}) - \psi_i(\mathbf{r}) + \Delta(\mathbf{r}) . \quad (6.61)$$

Hence, as successive iterations improve on $\psi_i(\mathbf{r})$ so that it is closer to $\psi(\mathbf{r})$, $\psi_s(\mathbf{r})$ is swamped by noise. Other variants of the Born iterative method are more robust and also less time-consuming, especially in higher dimensions.

7 References and further reading

References

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A good book for complex methods, and specifically also topics of relevance to wave scattering, such as asymptotic approximations and the Wiener-Hopf method, is:

M.J. Ablowitz and A.S. Fokas *Complex variables*, CUP, 1997

More on asymptotic Methods: Hinch *Perturbation Methods*, Cambridge: CUP 1991

The following book gives a very useful compendium of the main known solutions to 'canonical' scattering problems (including the asymptotic solutions) for both acoustic and electromagnetic waves:

J.J. Bowman, T.B.A. Senior and P.L.E Uslenghi *Electromagnetic and acoustic scattering by simple shapes*, North-Holland, 1969