Densities and Distributions

Back in section 12.1 I presented a careful and full definition of the word "function." This is useful even though you should already have a pretty good idea of what the word means. If you haven't read that section, now would be a good time. The reason to review it is that this definition doesn't handle all the cases that naturally occur. This will lead to the idea of a "generalized function."

There are (at least) two approaches to this subject. One that relates it to the ideas of functionals as you saw them in the calculus of variations, and one that is more intuitive and is good enough for most purposes. The latter appears in section 17.5, and if you want to jump there first, I can't stop you.

17.1 Density

What *is* density? If the answer is "mass per unit volume" then what does that mean? It clearly doesn't mean what it says, because you aren't required* to use a cubic meter.

It's a derivative. Pick a volume ΔV and find the mass in that volume to be Δm. The average volume-mass-density in that volume is $\Delta m/\Delta V$. If the volume is the room that you're sitting in, the mass includes you and the air and everything else in the room. Just as in defining the concept of velocity (instantaneous velocity), you have to take a limit. Here the limit is

$$\lim_{\Delta V\to 0}\frac{\Delta m}{\Delta V}=\frac{dm}{dV} \tag{17.1}$$

Even this isn't quite right, because the volume could as easily shrink to zero by approaching a line, and that's not what you want. It has to shrink to a point, but the standard notation doesn't let me say that without introducing more symbols than I want.

Of course there are other densities. If you want to talk about paper or sheet metal you may find area-mass-density to be more useful, replacing the volume ΔV by an area ΔA. Maybe even linear mass density if you are describing wires, so that the denominator is $\Delta\ell$. And why is the numerator a mass? Maybe you are describing volume-charge-density or even population density (people per area). This last would appear in mathematical notation as dN/dA.

This last example manifests a subtlety in all of these definitions. In the real world, you can't take the limit as $\Delta A\to 0$. When you count the number of people in an area you can't very well let the area shrink to zero. When you describe mass, remember that the world is made of atoms. If you let the volume shrink too much you'll either be between or inside the atoms. Maybe you will hit a nucleus; maybe not. This sort of problem means that you have to stop short of the mathematical limit and let the volume shrink to some size that still contains many atoms, but that is small enough so the quotient $\Delta m/\Delta V$ isn't significantly affected by further changing ΔV. Fortunately, this fine point seldom gets in the way, and if it does, you'll know it fast. I'll ignore it. If you're bothered by it remember that you are accustomed to doing the same thing when you approximate a sum by an integral. The world is made of atoms, and any common computation about a physical system will really involve a sum over all the atoms in the system (*e.g.* find the center of mass). You never do this, preferring to do an integral instead even though this is an approximation to the sum over atoms.

If you know the density — when the word is used unqualified it commonly means volume-mass-density — you find mass by integration over the volume you have specified.

$$m=\int_V \rho\, dV \tag{17.2}$$

* even by the panjandrums of the Système International d'Unités

You can even think of this as a new kind of function $m(V)$: input a specification for a volume of space; output a mass. That's really what density provides, a prescription to go from a volume specification to the amount of mass within that volume.

For the moment, I'll restrict the subject to linear mass density, and so that you simply need the coordinate along a straight line,

$$\lambda(x) = \frac{dm}{dx}(x), \qquad \text{and} \qquad m = \int_a^b \lambda(x)\,dx \tag{17.3}$$

If λ represents a function such as Ax^2 $(0 < x < L)$, a bullwhip perhaps, then this is elementary, and $m_{\text{total}} = AL^3/3$. I want to look at the reverse specification. Given an interval, I will specify the amount of mass in that interval and work backwards. The first example will be simple. The interval $x_1 \le x \le x_2$ is denoted $[x_1, x_2]$. The function m has this interval for its argument.*

$$m\big([x_1,x_2]\big) = \begin{cases} 0 & (x_1 \le x_2 \le 0) \\ Ax_2^3/3 & (x_1 \le 0 \le x_2 \le L) \\ AL^3/3 & (x_1 \le 0 \le L \le x_2) \\ A\big(x_2^3 - x_1^3\big)/3 & (0 \le x_1 \le x_2 \le L) \\ A\big(L^3 - x_1^3\big)/3 & (0 \le x_1 \le L \le x_2) \\ 0 & (L \le x_1 \le x_2) \end{cases} \tag{17.4}$$

The density Ax^2 $(0 < x < L)$ is of course a much easier way to describe the same distribution of mass. This distribution function, $m\big([x_1, x_2]\big)$, comes from integrating the density function $\lambda(x) = Ax^2$ on the interval $[x_1, x_2]$.

Another example is a variation on the same theme. It is slightly more involved, but still not too bad.

$$m\big([x_1,x_2]\big) = \begin{cases} 0 & (x_1 \le x_2 \le 0) \\ Ax_2^3/3 & (x_1 \le 0 \le x_2 < L/2) \\ Ax_2^3/3 + m_0 & (x_1 \le 0 < L/2 \le x_2 \le L) \\ AL^3/3 + m_0 & (x_1 \le 0 < L \le x_2) \\ A\big(x_2^3 - x_1^3\big)/3 & (0 \le x_1 \le x_2 < L/2) \\ A\big(x_2^3 - x_1^3\big)/3 + m_0 & (0 \le x_1 < L/2 \le x_2 \le L) \\ A\big(L^3 - x_1^3\big)/3 + m_0 & (0 \le x_1 \le L/2 < Ll2) \\ A\big(x_2^3 - x_1^3\big)/3 & (L/2 < x_1 \le x_2 \le L) \\ A\big(L^3 - x_1^3\big)/3 & (L/2 < x_1 \le L \le x_2) \\ 0 & (L \le x_1 \le x_2) \end{cases} \tag{17.5}$$

If you read through all these cases, you will see that the sole thing that I've added to the first example is a point mass m_0 at the point $L/2$. What density function λ will produce this distribution? Answer: *No function will do this*. That's why the concept of a "generalized function" appeared. I could state this distribution function in words by saying

"Take Eq. (17.4) and if $[x_1, x_2]$ contains the point $L/2$ then add m_0."

That there's no density function λ that will do this is inconvenient but not disastrous. When the very idea of a density was defined in Eq. (17.1), it started with the distribution function, the mass within the volume, and only arrived at the definition of a density by some manipulations. The density is a type of derivative and not all functions are differentiable. The function $m\big([x_1, x_2]\big)$ or $m(V)$ is more fundamental (if less convenient) than is the density function.

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* I'm abusing the notation here. In (17.2) m is a number. In (17.4) m is a function. You're used to this, and physicists do it all the time despite reproving glances from mathematicians.

17.2 Functionals

$$F[\phi]=\int_{-\infty}^{\infty}dx\,f(x)\phi(x)$$

defines a scalar-valued function of a function variable. Given any (reasonable) function ϕ as input, it returns a scalar. That is a functional. This one is a linear functional because it satisfies the equations

$$F[a\phi]=aF[\phi]\qquad\text{and}\qquad F[\phi_1+\phi_2]=F[\phi_1]+F[\phi_2] \tag{17.6}$$

This isn't a new idea, it's just a restatement of a familiar idea in another language. The mass density can define a useful functional (*linear* density for now). Given $dm/dx=\lambda(x)$ what is the total mass?

$$\int_{-\infty}^{\infty}dx\,\lambda(x)1=M_{\text{total}}$$

Where is the center of mass?

$$\frac{1}{M_{\text{total}}}\int_{-\infty}^{\infty}dx\,\lambda(x)x=x_{\text{cm}}$$

Restated in the language of functionals,

$$F[\phi]=\int_{-\infty}^{\infty}dx\,\lambda(x)\phi(x)\qquad\text{then}\qquad M_{\text{total}}=F[1],\qquad x_{\text{cm}}=\frac{1}{M_{\text{total}}}F[x]$$

If, instead of mass density, you are describing the distribution of grades in a large class or the distribution of the speed of molecules in a gas, there are still other ways to use this sort of functional. If $dN/dg=f(g)$ is the grade density in a class (number of students per grade interval), then with $F[\phi]=\int dg\,f(g)\phi(g)$

$$\begin{gathered}N_{\text{students}}=F[1],\qquad\text{mean grade}=\bar g=\frac{1}{N_{\text{students}}}F[g], \\ \text{variance}=\sigma^2=\frac{1}{N_{\text{students}}}F[(g-\bar g)^2],\qquad\text{skewness}=\frac{1}{N_{\text{students}}\sigma^3}F[(g-\bar g)^3] \\ \text{kurtosis excess}=\frac{1}{N_{\text{students}}\sigma^4}F[(g-\bar g)^4]-3\end{gathered} \tag{17.7}$$

Unless you've studied some statistics, you will probably never have heard of skewness and kurtosis excess. They are ways to describe the shape of the density function, and for a Gaussian both these numbers are zero. If it's skewed to one side the skewness is non-zero. [Did I really say that?] The kurtosis excess compares the flatness of the density function to that of a Gaussian.

The Maxwell-Boltzmann function describing the speeds of molecules in an ideal gas is at temperature T

$$f_{\text{MB}}(v)=\left(\frac{m}{2\pi kT}\right)^{3/2}4\pi v^2e^{-mv^2/2kT} \tag{17.8}$$

dN/dv is the number of molecules per speed interval, but this function F_{MB} is normalized differently. It is instead $(dN/dv)/N_{\text{total}}$. It is the fraction of the molecules per speed interval. That saves carrying along a factor specifying the total number of molecules in the gas. I could have done the same thing in the preceding example of student grades, defining the functional $F_1=F/N_{\text{students}}$. Then the equations

(17.7) would have a simpler appearance, such as $\bar{g} = F_1[g]$. For the present case of molecular speeds, with $F[\phi] = \int_0^\infty dv\, f_{\mathsf{MB}}(v)\phi(v)$,

$$F[1] = 1, \qquad F[v] = \bar{v} = \sqrt{\frac{8kT}{\pi m}}, \qquad F[mv^2/2] = \overline{\mathsf{K.E.}} = \frac{3}{2}kT \tag{17.9}$$

Notice that the mean kinetic energy is not the kinetic energy that corresponds to the mean speed.

Look back again to section 12.1 and you'll see not only a definition of "function" but a definition of "functional." It looks different from what I've been using here, but look again and you will see that when you view it in the proper light, that of chapter six, they are the same. Equations (12.3)–(12.5) involved vectors, but remember that when you look at them as elements of a vector space, functions are vectors too.

Functional Derivatives

In section 16.2, equations (16.6) through (16.10), you saw a development of the functional derivative. What does that do in this case?

$$F[\phi] = \int dx\, f(x)\phi(x), \qquad \text{so} \qquad F[\phi + \delta\phi] - F[\phi] = \int dx\, f(x)\delta\phi(x)$$

The functional derivative is the coefficient of $\delta\phi$ and dx, so it is

$$\frac{\delta F}{\delta\phi} = f \tag{17.10}$$

That means that the functional derivative of $m\big([x_1, x_2]\big)$ in Eq. (17.4) is the linear mass density, $\lambda(x) = Ax^2$, $(0 < x < L)$. Is there such a thing as a functional integral? Yes, but not here, as it goes well beyond the scope of this chapter. Its development is central in quantum field theory.

17.3 Generalization

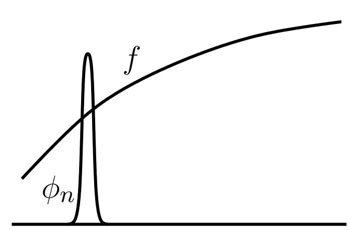
Given a function f, I can create a linear functional F using it as part of an integral. What sort of linear functional arises from f'? Integrate by parts to find out. Here I'm going to have to assume that f or ϕ or both vanish at infinity, or the development here won't work.

$$F[\phi] = \int_{-\infty}^{\infty} dx\, f(x)\phi(x), \qquad \text{then}$$

$$\int_{-\infty}^{\infty} dx f'(x)\phi(x) = f(x)\phi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\, f(x)\phi'(x) = -F[\phi'] \tag{17.11}$$

In the same way, you can relate higher derivatives of f to the functional F. There's another restriction you need to make: For this functional $-F[\phi']$ to make sense, the function ϕ has to be differentiable. If you want higher derivatives of f, then ϕ needs to have still higher derivatives.

If you know everything about F, what can you determine about f? If you assume that all the functions you're dealing with are smooth, having as many derivatives as you need, then the answer is simple: *everything*. If I have a rule by which to get a number $F[\phi]$ for every (smooth) ϕ, then I can take a special case for ϕ and use it to find f. Use a ϕ that drops to zero very rapidly away from some given point; for example if n is large this function drops off rapidly away from the point x_0.



$$\phi_n(x) = \sqrt{\frac{n}{\pi}} e^{-n(x-x_0)^2}$$

I've also arranged so that its integral of ϕ_n over all x is one. If I want the value of f at x_0 I can do an integral and take a limit.

$$F[\phi_n] = \int_{-\infty}^{\infty} dx\, f(x)\phi_n(x) = \int_{-\infty}^{\infty} dx\, f(x)\sqrt{\frac{n}{\pi}}e^{-n(x-x_0)^2}$$

As n increases to infinity, all that matters for f is its value at x_0. The integral becomes

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} dx\, f(x_0)\sqrt{\frac{n}{\pi}}e^{-n(x-x_0)^2} = f(x_0) \tag{17.12}$$

This means that I can reconstruct the function f if I know everything about the functional F. To get the value of the derivative $f'(x_0)$ instead, simply use the function $-\phi_n'$ and take the same limit. This construction is another way to look at the functional derivative. The equations (17.10) and (17.12) produce the same answer.

You say that this doesn't sound very practical? That it is an awfully difficult and roundabout way to do something? Not really. It goes back to the ideas of section 17.1. To define a density you have to know how much mass is contained in an arbitrarily specified volume. That's a functional. It didn't look like a functional there, but it is. You just have to rephrase it to see that it's the same thing.

As before, do the case of linear mass density. $\lambda(x)$ is the density, and the functional $F[\phi] = \int_{-\infty}^{\infty} dx\,\lambda(x)\phi(x)$. Then as in Eq. (17.4), $m\big([x_1,x_2]\big)$ is that mass contained in the interval from x_1 to x_2 and you can in turn write it as a functional.

$$\text{Let } \chi \text{ be the step function} \quad \chi(x) = \left\{\begin{matrix} 1 & (x_1 \le x \le x_2) \\ 0 & \text{(otherwise)}\end{matrix}\right\}$$

$$\text{then} \qquad \int_{x_1}^{x_2} dx\,\lambda(x) = F[\chi] = m\big([x_1,x_2]\big)$$

What happens if the function f is itself not differentiable? I can still define the original functional. Then I'll see what implications I can draw from the functional itself. The first and most important example of this is for f a step function.

$$f(x) = \theta(x) = \begin{cases} 1 & (x \ge 0) \\ 0 & (x < 0)\end{cases} \qquad F[\phi] = \int_{-\infty}^{\infty} dx\,\theta(x)\phi(x) \tag{17.13}$$

θ has a step at the origin, so of course it's not differentiable there, but if it were possible in some way to define its derivative, then when you look at its related functional, it should give the answer $-F[\phi']$ as in Eq. (17.11), just as for any other function. What then is $-F[\phi']$?

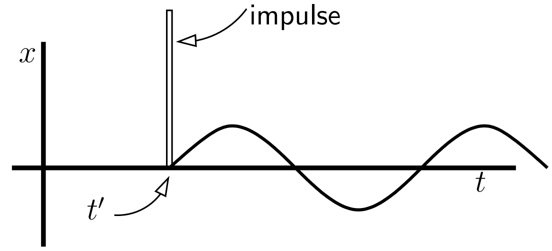
$$-F[\phi'] = -\int_{-\infty}^{\infty} dx\,\theta(x)\phi'(x) = -\int_{0}^{\infty} dx\,\phi'(x) = -\phi(x)\Big|_0^{\infty} = \phi(0) \tag{17.14}$$

This defines a perfectly respectable linear functional. Input the function ϕ and output the value of the function at zero. It easily satisfies Eq. (17.6), but there is no function f that when integrated against ϕ will yield this result. Still, it is so useful to be able to do these manipulations as if such a function exists that the notation of a "delta function" was invented. This is where the idea of a generalized function enters.

Green's functions
In the discussion of the Green's function solution to a differential equation in section 4.6, I started with the differential equation

$$m\ddot{x} + kx = F(t)$$

and found a general solution in Eq. (4.34). This approach pictured the external force as a series of small impulses and added the results from each.



$$x(t) = \int_{-\infty}^{\infty} dt'\, G(t-t')F(t') \qquad \text{where} \qquad G(t) = \begin{cases} \frac{1}{m\omega_0}\sin\omega_0 t & (t \ge 0) \\ 0 & (t < 0) \end{cases} \tag{17.15}$$

I wrote it a little differently there, but it's the same. Can you verify that this is a solution to the stated differential equation? Simply plug in, do a couple of derivatives and see what happens. One derivative at a time:

$$\frac{dx}{dt} = \int_{-\infty}^{\infty} dt'\, \dot{G}(t-t')F(t') \qquad \text{where} \qquad \dot{G}(t) = \begin{cases} \frac{1}{m}\cos\omega_0 t & (t \ge 0) \\ 0 & (t < 0) \end{cases} \tag{17.16}$$

Now for the second derivative. *Oops.* I can't differentiate $\dot{G}$. It has a step at $t = 0$.

This looks like something you saw in a few paragraphs back, where there was a step in the function θ. I'm going to handle this difficulty now by something of a kludge. In the next section you'll see the notation that makes this manipulation easy and transparent. For now I will subtract and add the discontinuity from $\dot{G}$ by using the same step function θ.

$$\begin{aligned} \dot{G}(t) &= \begin{cases} \frac{1}{m}\big[\cos\omega_0 t - 1 + 1\big] & (t \ge 0) \\ 0 & (t < 0) \end{cases} \\ &= \left.\begin{cases} \frac{1}{m}\big[\cos\omega_0 t - 1\big] & (t \ge 0) \\ 0 & (t < 0) \end{cases}\right\} + \frac{1}{m}\theta(t) = \dot{G}_0(t) + \frac{1}{m}\theta(t) \end{aligned} \tag{17.17}$$

The (temporary) notation here is that $\dot{G}_0$ is the part of $\dot{G}$ that doesn't have the discontinuity at $t = 0$. That part is differentiable. The expression for dx/dt now has two terms, one from the $\dot{G}_0$ and one from the θ. Do the first one:

$$\frac{d}{dt}\int_{-\infty}^{\infty} dt'\, \dot{G}_0(t-t')F(t') = \int_{-\infty}^{\infty} dt'\, \frac{d}{dt}\dot{G}_0(t-t')F(t')$$

$$\text{and} \qquad \frac{d}{dt}\dot{G}_0(t) = \begin{cases} \frac{1}{m}[-\omega_0\sin\omega_0 t] & (t \ge 0) \\ 0 & (t < 0) \end{cases}$$

The original differential equation involved $m\ddot{x} + kx$. The $\dot{G}_0$ part of this is

$$m\int_{-\infty}^{\infty} dt' \left.\begin{cases} \frac{1}{m}[-\omega_0\sin\omega_0(t-t')] & (t \ge t') \\ 0 & (t < t') \end{cases}\right\} F(t') \\ + k\int_{-\infty}^{\infty} dt' \left.\begin{cases} \frac{1}{m\omega_0}\sin\omega_0(t-t') & (t \ge t') \\ 0 & (t < t') \end{cases}\right\} F(t')$$

Use $k = m\omega_0^2$, and this is zero.

Now go back to the extra term in θ. The kx terms doesn't have it, so all that's needed is

$$m\ddot x + kx = m\frac{d}{dt}\int_{-\infty}^{\infty} dt'\,\frac{1}{m}\theta(t-t')F(t') = \frac{d}{dt}\int_{-\infty}^{t} dt'\,F(t') = F(t)$$

This verifies yet again that this Green's function solution works.

17.4 Delta-function Notation

Recognizing that it would be convenient to be able to differentiate non-differentiable functions, that it would make manipulations easier if you could talk about the density of a point mass ($m/0 = ?$), and that the impulsive force that appears in setting up Green's functions for the harmonic oscillator isn't a function, what do you do? If you're Dirac, you invent a notation that works.

Two functionals are equal to each other if they give the same result for all arguments. Does that apply to the functions that go into defining them? If

$$\int_{-\infty}^{\infty} dx\,f_1(x)\phi(x) = \int_{-\infty}^{\infty} dx\,f_2(x)\phi(x)$$

for all test functions ϕ (smooth, infinitely differentiable, going to zero at infinity, whatever constraints we find expedient), does it mean that $f_1 = f_2$? Well, no. Suppose that I change the value of f_1 at exactly one point, adding 1 there and calling the result f_2. These functions aren't equal, but underneath an integral sign you can't tell the difference. In terms of the functionals they define, they are essentially equal: "equal in the sense of distributions."

Extend this to the generalized functions

The equation (17.14) leads to specifying the functional

$$\delta[\phi] = \phi(0) \tag{17.18}$$

This delta-functional isn't a help in doing manipulations, so define the notation

$$\int_{-\infty}^{\infty} dx\,\delta(x)\phi(x) = \delta[\phi] = \phi(0) \tag{17.19}$$

This notation isn't an integral in the sense of something like section 1.6, and $\delta(x)$ isn't a function, but the notation allows you effect manipulations just as if they were. Note: the symbol δ here is not the same as the δ in the functional derivative. We're just stuck with using the same symbol for two different things. Blame history and look at problem 17.10.

You can treat the step function as differentiable, with $\theta' = \delta$, and this notation leads you smoothly to the right answer.

$$\text{Let}\quad \theta_{x_0}(x) = \begin{cases} 1 & (x \ge x_0) \\ 0 & (x < x_0) \end{cases} = \theta(x - x_0), \quad \text{then}\quad \int_{-\infty}^{\infty} dx\,\theta_{x_0}(x)\phi(x) = \int_{x_0}^{\infty} dx\,\phi(x)$$

The derivative of this function is

$$\frac{d}{dx}\theta_{x_0}(x) = \delta(x - x_0)$$

You show this by

$$\int_{-\infty}^{\infty} dx\, \frac{d\theta_{x_0}(x)}{dx}\phi(x) = -\int_{-\infty}^{\infty} dx\, \theta_{x_0}(x)\phi'(x) = -\int_{x_0}^{\infty} dx\, \phi'(x) = \phi(x_0)$$

The idea of a generalized function is that you can manipulate it *as if* it were an ordinary function provided that you put the end results of your manipulations under an integral.

The above manipulations for the harmonic oscillator, translated to this language become

$$m\ddot{G} + kG = \delta(t) \qquad \text{for} \qquad G(t) = \begin{cases} \frac{1}{m\omega_0}\sin\omega_0 t & (t \ge 0) \\ 0 & (t < 0) \end{cases}$$

Then the solution for a forcing function $F(t)$ is

$$x(t) = \int_{-\infty}^{\infty} G(t-t')F(t')\,dt'$$

because

$$m\ddot{x} + kx = \int_{-\infty}^{\infty} (m\ddot{G} + kG)F(t')\,dt' = \int_{-\infty}^{\infty} \delta(t-t')F(t')\,dt' = F(t)$$

This is a lot simpler. Is it legal? Yes, though it took some serious mathematicians (Schwartz, Sobolev) some serious effort to develop the logical underpinnings for this subject. The result of their work is: It's o.k.

17.5 Alternate Approach

This delta-function method is so valuable that it's useful to examine it from more than one vantage. Here is a very different way to understand delta functions, one that avoids an explicit discussion of functionals. Picture a sequence of smooth functions that get narrower and taller as the parameter n gets bigger. Examples are

$$\sqrt{\frac{n}{\pi}}e^{-nx^2}, \qquad \frac{n}{\pi}\frac{1}{1+n^2x^2}, \qquad \frac{1}{\pi}\frac{\sin nx}{x}, \qquad \frac{n}{\pi}\operatorname{sech} nx \tag{17.20}$$

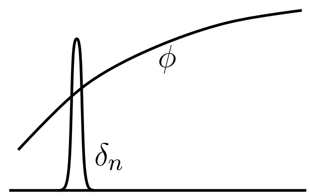
Pick any one such sequence and call it $\delta_n(x)$. (A "delta sequence") The factors in each case are arranged so that

$$\int_{-\infty}^{\infty} dx\, \delta_n(x) = 1$$

As n grows, each function closes in around $x = 0$ and becomes very large there. Because these are perfectly smooth functions there's no question about integrating them.

$$\int_{-\infty}^{\infty} dx\, \delta_n(x)\phi(x) \tag{17.21}$$

makes sense as long as ϕ doesn't cause trouble. You will typically have to assume that the ϕ behave nicely at infinity, going to zero fast enough, and this is satisfied in the physics applications that we need. For large n any of these functions looks like a very narrow spike. If you multiply one of these δ_ns by a mass m, you have a linear mass density that is (for large n) concentrated near to a point: $\lambda(x) = m\delta_n(x)$. Of course you can't take the limit as $n \to \infty$ because this doesn't have a limit. If you could, then that would be the density for a point mass: $m\delta(x)$.



What happens to (17.21) as $n\to\infty$? For large n any of these delta-sequences approaches zero everywhere except at the origin. Near the origin $\phi(x)$ is very close to $\phi(0)$, and the function δ_n is non-zero in only the tiny region around zero. If the function ϕ is simply continuous at the origin you have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} dx\,\delta_n(x)\phi(x) = \phi(0)\cdot\lim_{n\to\infty}\int_{-\infty}^{\infty} dx\,\delta_n(x) = \phi(0) \tag{17.22}$$

At this point I can introduce a notation:

$$\text{“}\int_{-\infty}^{\infty} dx\,\delta(x)\phi(x)\text{”} \qquad \textit{MEANS} \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty} dx\,\delta_n(x)\phi(x) \tag{17.23}$$

In this approach to distributions the collection of symbols on the left has for its definition the collection of symbols on the right. Those in turn, such as $\int$, have definitions that go back to the fundamentals of calculus. You *cannot* move the limit in this last integral under the integral sign. You can't interchange these limits because the limit of δ_n is not a function.

In this development you say that the delta function is a notation, not for a function, but for a process (but then, so is the integral sign). That means that the underlying idea always goes back to a familiar, standard manipulation of ordinary calculus. If you have an equation involving such a function, say

$$\theta'(x) = \delta(x), \qquad \text{then this } \textit{means} \qquad \theta_n'(x) = \delta_n(x) \qquad \text{and that}$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} dx\,\theta_n'(x)\phi(x) = \lim_{n\to\infty}\int_{-\infty}^{\infty} dx\,\delta_n(x)\phi(x) = \phi(0) \tag{17.24}$$

How can you tell if this equation is true? Remember now that in this interpretation these are sequences of ordinary, well-behaved functions, so you can do ordinary manipulations such as partial integration. The functions θ_n are smooth and they rise from zero to one as x goes from $-\infty$ to $+\infty$. As n becomes large, the interval over which this rise occurs will become narrower and narrower. In the end these θ_n will approach the step function $\theta(x)$ of Eq. (17.13).

$$\int_{-\infty}^{\infty} dx\,\theta_n'(x)\phi(x) = \theta_n(x)\phi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} dx\,\theta_n(x)\phi'(x)$$

The functions ϕ go to zero at infinity — they're “test functions” — and that kills the boundary terms, leaving the last integral standing by itself. Take the limit as $n\to\infty$ on it. You can take the limit inside the integral now, because the limit of θ_n is a perfectly good function, even if it is discontinuous.

$$-\lim_{n}\int_{-\infty}^{\infty} dx\,\theta_n(x)\phi'(x) = -\int_{-\infty}^{\infty} dx\,\theta(x)\phi'(x) = -\int_{0}^{\infty} dx\,\phi'(x) = -\phi(x)\Big|_{0}^{\infty} = \phi(0)$$

This is precisely what the second integral in Eq. (17.24) is. This is the proof that $\theta' = \delta$. Any proof of an equation involving such generalized functions requires you to integrate the equation against a test function and to determine if the resulting integral becomes an identity as $n\to\infty$. This implies that it now makes sense to differentiate a discontinuous function — as long as you mean differentiation “in the sense of distributions.” That's the jargon you encounter here. An equation such as $\theta' = \delta$ makes sense only when it is under an integral sign and is interpreted in the way that you just saw.

In these manipulations, where did I use the particular form of the delta sequence? Never. A particular combination such as

$$\theta_n(x) = \frac{1}{2}\big[1 + \frac{2}{\pi}\tan^{-1} nx\big], \qquad \text{and} \qquad \delta_n(x) = \frac{n}{\pi}\frac{1}{1+n^2x^2} \tag{17.25}$$

never appeared. Any of the other sequences would have done just as well, and all that I needed was the *properties* of the sequence, not its particular representation. You can even use a delta sequence that doesn't look like any of the functions in Eq. (17.20).

$$\delta_n(x) = 2\sqrt{\frac{2n}{\pi}}e^{inx^2} \tag{17.26}$$

This turns out to have all the properties that you need, though again, you don't have to invoke its explicit form.

What is $\delta(ax)$? Integrate $\delta_n(ax)$ with a test function.

$$\lim_n \int_{-\infty}^{\infty} dx\, \delta_n(ax)\phi(x) = \lim_n \int_{-\infty}^{\infty} \frac{dy}{a}\, \delta_n(y)\phi(y/a)$$

where $y = ax$. Actually, this isn't quite right. If $a > 0$ it is fine, but if a is negative, then when $x \to -\infty$ you have $y \to +\infty$. You have to change the limits to put it in the standard form. You can carry out that case for yourself and verify that the expression covering both cases is

$$\lim_n \int_{-\infty}^{\infty} dx\, \delta_n(ax)\phi(x) = \lim_n \int_{-\infty}^{\infty} \frac{dy}{|a|}\, \delta_n(y)\phi(y/a) = \frac{1}{|a|}\phi(0)$$

Translate this into the language of delta functions and it is

$$\delta(ax) = \frac{1}{|a|}\delta(x) \tag{17.27}$$

You can prove other relations in the same way. For example

$$\delta(x^2 - a^2) = \frac{1}{2|a|}\big[\delta(x-a) + \delta(x+a)\big] \qquad \text{or} \qquad \delta\big(f(x)\big) = \sum_k \frac{1}{|f'(x_k)|}\delta(x - x_k) \tag{17.28}$$

In the latter equation, x_k is a root of f, and you sum over all roots. Notice that it doesn't make any sense if you have a double root. Just try to see what $\delta(x^2)$ would mean. The last of these identities contains the others as special cases. Eq. (17.27) implies that δ is even.

17.6 Differential Equations

Where do you use these delta functions? Start with differential equations. I'll pick one that has the smallest number of technical questions associated with it. I want to solve the equation

$$\frac{d^2 f}{dx^2} - k^2 f = F(x) \tag{17.29}$$

subject to conditions that $f(x)$ should approach zero for large magnitude x. I'll assume that the given function F has this property too.

But first: Let k and y be constants, then solve

$$\frac{d^2g}{dx^2} - k^2 g = \delta(x-y) \tag{17.30}$$

I want a solution that is well-behaved at infinity, staying finite there. This equality is "in the sense of distributions" recall, and I'll derive it a couple of different ways, here and in the next section.

First way: Treat the δ as simply a spike at $x = y$. Everywhere else on the x-axis it is zero. Solve the equation for two cases then, $x < y$ and $x > y$. In both cases the form is the same.

$$g'' - k^2 g = 0, \qquad \text{so} \qquad g(x) = Ae^{kx} + Be^{-kx}$$

For $x < y$, I want $g(x)$ to go to zero far away, so that requires the coefficient of e^{-kx} to be zero. For $x > 0$, the reverse is true and only the e^{-kx} can be present.

$$g(x) = \begin{cases} Ae^{kx} & (x < y) \\ Be^{-kx} & (x > y) \end{cases}$$

Now I have to make g satisfy Eq. (17.30) *at* $x = y$.

Compute dg/dx. But wait. This is impossible unless the function is at least continuous. If it isn't then I'd be differentiating a step function and I don't want to do that (at least not yet). That is

$$g(y-) = Ae^{ky} = g(y+) = Be^{-ky} \tag{17.31}$$

This is one equation in the two unknowns A and B. Now differentiate.

$$\frac{dg}{dx} = \begin{cases} Ake^{kx} & (x < y) \\ -Bke^{-kx} & (x > y) \end{cases} \tag{17.32}$$

This is in turn differentiable everywhere except at $x = y$. There it has a step

$$\text{discontinuity in } g' = g'(y+) - g'(y-) = -Bke^{-ky} - Ake^{ky}$$

This means that (17.32) is the sum of two things, one differentiable, and the other a step, a multiple of θ.

$$\frac{dg}{dx} = \text{differentiable stuff} + \big(-Bke^{-ky} - Ake^{ky}\big)\theta(x-y)$$

The differentiable stuff satisfies the differential equation $g'' - k^2 g = 0$. For the rest, Compute d^2g/dx^2

$$\big(-Bke^{-ky} - Ake^{ky}\big)\frac{d}{dx}\theta(x-y) = \big(-Bke^{-ky} - Ake^{ky}\big)\delta(x-y)$$

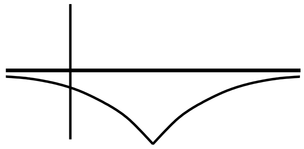
Put this together and remember the equation you're solving, Eq. (17.30).

$$\begin{aligned} &g'' - k^2 g = 0 \text{ (from the differentiable stuff)} \\ &\qquad + \big(-Bke^{-ky} - Ake^{ky}\big)\delta(x-y) \quad = \delta(x-y) \end{aligned}$$

Now there are two equations for A and B, this one and Eq. (17.31).

$$\begin{aligned} Ae^{ky} &= Be^{-ky} \\ -Bke^{-ky} - Ake^{ky} &= 1 \end{aligned} \qquad \text{solve these to get} \qquad \begin{aligned} A &= -e^{-ky}/2k \\ B &= -e^{ky}/2k \end{aligned}$$

Finally, back to g.



$$g(x) = \begin{cases} -e^{k(x-y)}/2k & (x < y) \\ -e^{-k(x-y)}/2k & (x > y) \end{cases} \tag{17.33}$$

When you get a fairly simple form of solution such as this, you have to see if you could have saved some work, perhaps replacing brute labor with insight? Of course. The original differential equation (17.30) is symmetric around the point y. It's plausible to look for a solution that behaves the same way, using $(x - y)$ as the variable instead of x. Either that or you could do the special case $y = 0$ and then change variables at the end to move the delta function over to y. See problem 17.11.

There is standard notation for this function; it *is* a Green's function.

$$G(x, y) = \left.\begin{cases} -e^{k(x-y)}/2k & (x < y) \\ -e^{-k(x-y)}/2k & (x > y) \end{cases}\right\} = -e^{-k|x-y|}/2k \tag{17.34}$$

Now to solve the original differential equation, Eq. (17.29). Substitute into this equation the function

$$\int_{-\infty}^{\infty} dy\, G(x, y)F(y)$$

$$\begin{aligned}\left[\frac{d^2}{dx^2} - k^2\right] \int_{-\infty}^{\infty} dy\, G(x, y)F(y) &= \int_{-\infty}^{\infty} dy\, \left[\frac{d^2 G(x, y)}{dx^2} - k^2 G(x, y)\right] F(y) \\ &= \int_{-\infty}^{\infty} dy\, \delta(x - y)F(y) = F(x)\end{aligned}$$

This is the whole point of delta functions. They make this sort of manipulation as easy as dealing with an ordinary function. Easier, once you're used to them.

For example, if $F(x) = F_0$ between $-x_0$ and $+x_0$ and zero elsewhere, then the solution for f is

$$\begin{aligned}\int dy\, G(x, y)F(y) &= -\int_{-x_0}^{x_0} dy\, F_0 e^{-k|x-y|}/2k \\ &= -\frac{F_0}{2k} \begin{cases} \int_{-x_0}^{x_0} dy\, e^{-k(x-y)} & (x > x_0) \\ \int_{-x_0}^{x} dy\, e^{-k(x-y)} + \int_{x}^{x_0} dy\, e^{-k(y-x)} & (-x_0 < x < x_0) \\ \int_{-x_0}^{x_0} dy\, e^{-k(y-x)} & (x < -x_0) \end{cases} \\ &= -\frac{F_0}{k^2} \begin{cases} e^{-kx} \sinh kx_0 & (x > x_0) \\ [1 - e^{-kx_0} \cosh kx] & (-x_0 < x < x_0) \\ e^{kx} \sinh kx_0 & (x < -x_0) \end{cases}\end{aligned} \tag{17.35}$$

You can see that this resulting solution is an even function of x, necessarily so because the original differential equation is even in x, the function $F(x)$ is even in x, and the boundary conditions are even in x.

Other Differential Equations

Can you apply this method to other equations? Yes, many. Try the simplest first order equation:

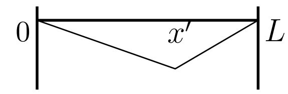
$$\begin{aligned}\frac{dG}{dx} &= \delta(x) \longrightarrow G(x) = \theta(x) \\ \frac{df}{dx} &= g(x) \longrightarrow f(x) = \int_{-\infty}^{\infty} dx' G(x - x')g(x') = \int_{-\infty}^{x} dx'\, g(x')\end{aligned}$$

which clearly satisfies $df/dx = g$.

If you try $d^2G/dx^2 = \delta(x)$ you explain the origin of problem 1.48.

Take the same equation $d^2G/dx^2 = \delta(x - x')$ but in the domain $0 < x < L$ and with the boundary conditions $G(0) = 0 = G(L)$. The result is

$$G(x) = \begin{cases} x(x' - L)/L & (0 < x < x') \\ x'(x - L)/L & (x' < x < L) \end{cases} \tag{17.36}$$



17.7 Using Fourier Transforms

Solve Eq. (17.30) another way. Fourier transform everything in sight.

$$\frac{d^2g}{dx^2} - k^2g = \delta(x - y) \;\to\; \int_{-\infty}^{\infty} dx\,\left[\frac{d^2g}{dx^2} - k^2g\right]e^{-iqx} = \int_{-\infty}^{\infty} dx\,\delta(x - y)e^{-iqx} \tag{17.37}$$

The right side is designed to be easy. For the left side, do some partial integrations. I'm looking for a solution that goes to zero at infinity, so the boundary terms will vanish. See Eq. (15.12).

$$\int_{-\infty}^{\infty} dx\,\big[-q^2 - k^2\big]g(x)e^{-iqx} = e^{-iqy} \tag{17.38}$$

The left side involves only the Fourier transform of g. Call it $\tilde g$.

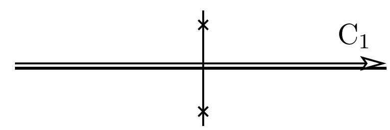
$$\big[-q^2 - k^2\big]\tilde g(q) = e^{-iqy}, \qquad \text{so} \qquad \tilde g(q) = -\frac{e^{-iqy}}{q^2 + k^2}$$

Now invert the transform.

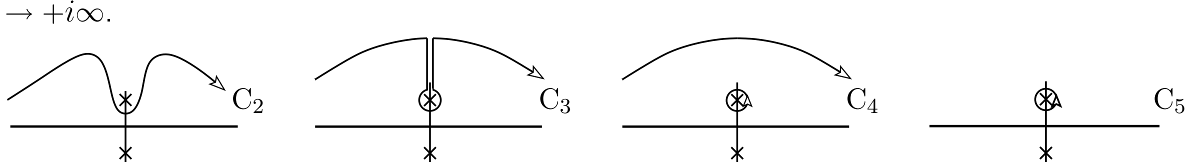
$$g(x) = \int_{-\infty}^{\infty} \frac{dq}{2\pi}\,\tilde g(q)e^{iqx} = -\int_{-\infty}^{\infty} \frac{dq}{2\pi}\,\frac{e^{iq(x-y)}}{q^2 + k^2}$$

Do this by contour integration, where the integrand has singularities at $q = \pm ik$.

$$-\frac{1}{2\pi}\int_{C_1} dq\,\frac{e^{iq(x-y)}}{k^2 + q^2}$$



The poles are at $\pm ik$, and the exponential dominates the behavior of the integrand at large $|q|$, so there are two cases: $x > y$ and $x < y$. Pick the first of these, then the integrand vanishes rapidly as $q \to +i\infty$.



$$\int_{C_1} = \int_{C_5} = \frac{-2\pi i}{2\pi}\,\operatorname*{Res}_{q=ik}\frac{e^{iq(x-y)}}{k^2 + q^2}$$

Compute the residue.

$$\frac{e^{iq(x-y)}}{k^2 + q^2} = \frac{e^{iq(x-y)}}{(q - ik)(q + ik)} \approx \frac{e^{iq(x-y)}}{(q - ik)(2ik)}$$

The coefficient of $1/(q - ik)$ is the residue, so

$$g(x) = -\frac{e^{-k(x-y)}}{2k} \qquad (x > y) \tag{17.39}$$

in agreement with Eq. (17.33). The $x < y$ case is yours.

17.8 More Dimensions

How do you handle problems in three dimensions? $\delta(\vec r\,) = \delta(x)\delta(y)\delta(z)$. For example I can describe the charge density of a point charge, dq/dV as $q\delta(\vec r - \vec r_0)$. The integral of this is

$$\int q\delta(\vec r - \vec r_0)\, d^3r = q$$

as long as the position $\vec r_0$ is inside the volume of integration. For an example that uses this, look at the potential of a point charge, satisfying Poisson's equation.

$$\nabla^2 V = -\rho/\epsilon_0$$

What is the potential for a specified charge density? Start by finding the Green's function.

$$\nabla^2 G = \delta(\vec r\,), \qquad \text{or instead do:} \qquad \nabla^2 G - k^2 G = \delta(\vec r\,) \tag{17.40}$$

The reason for starting with the latter equation is that you run into some problems with the first form. It's too singular. I'll solve the second one and then take the limit as $k \to 0$. The Fourier transform method is simpler, so use that.

$$\int d^3r \bigl[\nabla^2 G - k^2 G\bigr] e^{-i\vec q\,\cdot\,\vec r} = 1$$

When you integrate by parts (twice) along each of the three integration directions dx, dy, and dz, you pull down a factor of $-q^2 = -q_x^2 - q_y^2 - q_z^2$ just as in one dimension.

$$\int d^3r \bigl[-q^2 G - k^2 G\bigr] e^{-i\vec q\,\cdot\,\vec r} = 1 \qquad \text{or} \qquad \tilde G(\vec q\,) = \frac{-1}{q^2 + k^2}$$

where $\tilde G$ is, as before, the Fourier transform of G.

Now invert the transform. Each dimension picks up a factor of $1/2\pi$.

$$G(\vec r\,) = -\int \frac{d^3q}{(2\pi)^3} \frac{e^{i\vec q\,\cdot\,\vec r}}{q^2 + k^2}$$

This is a three dimensional integral, and the coordinate system to choose is spherical. q^2 doesn't depend on the direction of $\vec q$, and the single place that this direction matters is in the dot product in the exponent. The coordinates are q, θ, ϕ for the vector $\vec q$, and since I can pick my coordinate system any way I want, I will pick the coordinate axis along the direction of the vector $\vec r$. Remember that in this integral $\vec r$ is just some constant.

$$G = -\frac{1}{(2\pi)^3} \int q^2\, dq \sin\theta\, d\theta\, d\phi\, \frac{e^{iqr\cos\theta}}{q^2 + k^2}$$

The integral $d\phi$ becomes a factor 2π. Let $u = \cos\theta$, and that integral becomes easy. All that is left is the dq integral.

$$G = -\frac{1}{(2\pi)^2} \int_0^\infty \frac{q^2\, dq}{q^2 + k^2} \frac{1}{iqr} \bigl[e^{iqr} - e^{-iqr}\bigr]$$

More contour integrals: There are poles in the q-plane at $q = \pm ik$. The q^2 factors are even. The q in the denominator would make the integrand odd except that the combination of exponentials in brackets are also odd. The integrand as a whole is even. I will then extend the limits to $\pm\infty$ and divide by 2.

$$G = -\frac{1}{8\pi^2 ir}\int_{-\infty}^{\infty}\frac{q\,dq}{q^2+k^2}\big[e^{iqr} - e^{-iqr}\big]$$

There are two terms; start with the first one. $r > 0$, so $e^{iqr} \to 0$ as $q \to +i\infty$. The contour is along the real axis, so push it toward $i\infty$ and pick up the residue at $+ik$.

$$-\frac{1}{8\pi^2 ir}\int = -\frac{1}{8\pi^2 ir}2\pi i \operatorname*{Res}_{q=ik}\frac{q}{(q-ik)(q+ik)}e^{iqr} = -\frac{1}{8\pi^2 ir}2\pi i\frac{ik}{2ik}e^{-kr} \tag{17.41}$$

For the second term, see problem 17.15. Combine it with the preceding result to get

$$G = -\frac{1}{4\pi r}e^{-kr} \tag{17.42}$$

This is the solution to Eq. (17.40), and if I now let k go to zero, it is $G = -1/4\pi r$.

Just as in one dimension, once you have the Green's function, you can write down the solution to the original equation.

$$\nabla^2 V = -\rho/\epsilon_0 \quad\Longrightarrow\quad V = -\frac{1}{\epsilon_0}\int d^3r'\,G(\vec r,\vec r\,')\rho(\vec r\,') = \frac{1}{4\pi\epsilon_0}\int d^3r'\,\frac{\rho(\vec r\,')}{|\vec r - \vec r\,'|} \tag{17.43}$$

State this equation in English, and it says that the potential of a single point charge is $q/4\pi\epsilon_0 r$ and that the total potential is the sum over the contributions of all the charges. Of course this development also provides the Green's function for the more complicated equation (17.42).

Applications to Potentials

Let a charge density be $q\delta(\vec r)$. This satisfies $\int d^3r\,\rho = q$. The potential it generates is that of a point charge at the origin. This just reproduces the Green's function.

$$\phi = \frac{1}{4\pi\epsilon_0}\int d^3r'\,q\,\delta(\vec r\,')\frac{1}{|\vec r - \vec r\,'|} = \frac{q}{4\pi\epsilon_0 r} \tag{17.44}$$

What if $\rho(\vec r) = -p\,\partial\delta(\vec r)/\partial z$? [Why $-p$? Patience.] At least you can say that the dimensions of the constant p are charge times length. Now use the Green's function to compute the potential it generates.

$$\begin{aligned}\phi = \frac{1}{4\pi\epsilon_0}\int d^3r'\,(-p)\frac{\partial\delta(\vec r\,')}{\partial z'}\frac{1}{|\vec r - \vec r\,'|} &= \frac{p}{4\pi\epsilon_0}\int d^3r'\,\delta(\vec r\,')\frac{\partial}{\partial z'}\frac{1}{|\vec r - \vec r\,'|}\\ &= \frac{p}{4\pi\epsilon_0}\frac{\partial}{\partial z'}\frac{1}{|\vec r - \vec r\,'|}\bigg|_{\vec r\,'=0}\end{aligned} \tag{17.45}$$

This is awkward, so I'll use a little trick. Make a change of variables in (17.45) $\vec u = \vec r - \vec r\,'$, then

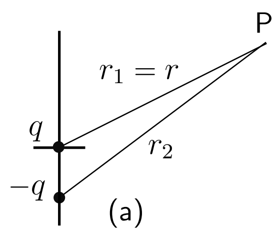
$$\frac{\partial}{\partial z'} \to -\frac{\partial}{\partial u_z}, \qquad\qquad \frac{p}{4\pi\epsilon_0}\frac{\partial}{\partial z'}\frac{1}{|\vec r - \vec r\,'|}\bigg|_{\vec r\,'=0} = -\frac{p}{4\pi\epsilon_0}\frac{\partial}{\partial u_z}\frac{1}{u} \tag{17.46}$$

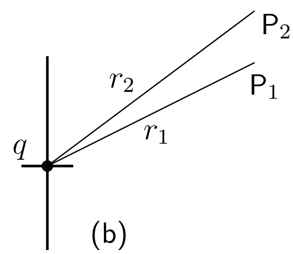
(See also problem 17.19.) Cutting through the notation, this last expression is just

$$\phi = \frac{-p}{4\pi\epsilon_0}\frac{\partial}{\partial z}\frac{1}{r} = \frac{-p}{4\pi\epsilon_0}\frac{\partial}{\partial z}\frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{p}{4\pi\epsilon_0}\frac{z}{\left(x^2+y^2+z^2\right)^{3/2}}$$
$$= \frac{p}{4\pi\epsilon_0}\frac{z}{r^3} = \frac{p}{4\pi\epsilon_0}\frac{\cos\theta}{r^2} \tag{17.47}$$

The expression $-\partial(1/r)/\partial z$ is such a simple way to compute this, the potential of an electric dipole, that it is worth trying to understand why it works. And in the process, *why* is this an electric dipole? The charge density, the source of the potential is a derivative, and a derivative is (the limit of) a difference quotient. This density is just that of two charges, and they produce potentials just as in Eq. (17.44). There are two point charges, with delta-function densities, one at the origin the other at $-\hat z\,\Delta z$.

$$\rho = \frac{-p}{\Delta z}\big[\delta(\vec r + \hat z\,\Delta z) - \delta(\vec r\,)\big] \qquad \text{gives potential} \qquad \frac{-p}{4\pi\epsilon_0\Delta z}\left[\frac{1}{|\vec r + \hat z\,\Delta z|} - \frac{1}{r}\right] \tag{17.48}$$





The picture of the potential that arises from this pair of charges is (a). A negative charge ($-q = -p/\Delta z$) at $-\hat z\,\Delta z$ and a corresponding positive charge $+q$ at the origin. This picture explains why this charge density, $\rho(\vec r\,) = -p\,\partial\delta(\vec r\,)/\partial z$, represent an electric dipole. Specifically it represents a dipole whose vector representation points toward $+\hat z$, from the negative charge to the positive one. It even explains why the result (17.47) has the sign that it does: The potential ϕ is positive along the positive z-axis and negative below. That's because a point on the positive z-axis is closer to the positive charge than it is to the negative charge.

Now the problem is to understand *why* the potential of this dipole ends up with a result as simple as the derivative $(-p/4\pi\epsilon_0)\partial(1/r)/\partial z$: The potential at the point P in the figure (a) comes from the two charges at the two distances r_1 and r_2. Construct figure (b) by moving the line r_2 upward so that it starts at the origin. Imagine a new charge configuration consisting of only *one* charge, $q = p/\Delta z$ at the origin. Now evaluate the potentials that this single charge produces at two different points P_1 and P_2 that are a distance Δz apart. Then subtract them.

$$\phi(\mathsf{P}_2) - \phi(\mathsf{P}_1) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

In the notation of the preceding equations this is

$$\phi(\mathsf{P}_2) - \phi(\mathsf{P}_1) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\vec r + \hat z\,\Delta z|} - \frac{1}{r}\right] = \frac{p}{4\pi\epsilon_0\Delta z}\left[\frac{1}{|\vec r + \hat z\,\Delta z|} - \frac{1}{r}\right]$$

Except for a factor of (-1), this is Eq. (17.48), and it explains why the potential caused by an ideal electric dipole at the origin can be found by taking the potential of a point charge and differentiating it.

$$\phi_{\text{point charge}} = \frac{q}{4\pi\epsilon_0 r} \qquad\qquad \phi_{\text{dipole}} = -a\frac{\partial}{\partial z}\phi_{\text{point charge}}$$

Here, the electric dipole strength is qa.

Can you repeat this process? Yes. Instead of two opposite point charges near each other, you can place two opposite point dipoles near each other.

$$\phi_{\text{linear quadrupole}} = -a\frac{\partial}{\partial z}\phi_{\text{dipole}} \tag{17.49}$$

This is the potential from the charge density $\rho = +Q\,\partial^2\delta(\vec r)/\partial z^2$, where $Q = qa^2$. [What about $\partial^2/\partial x\partial y$?]

Exercises

1 What is the analog of Eq. (17.4) for the linear mass density $\lambda(x) = C$ (a constant) for $0 < x < L$ and zero otherwise?

2 Take the preceding mass density and add a point mass m_0 at $x = L/2$. What is the distribution $m([x_1, x_2])$ now?

3 Use the λ from the first exercise and define the functional $F[\phi] = \int_{-\infty}^{\infty} dx\,\lambda(x)\phi(x)$. What is the total mass, $F[1] = M$? What is the mean position of the mass, $F[x]/M$?

4 As in the preceding exercise, what are the variance, the skewness, and the kurtosis excess?

5 What is $\int_0^1 dx\,\delta(x - x_0)$?

6 Pick any two of Eq. (17.20) and show that they are valid delta sequences.

7 What is $\int_{-\infty}^{x} dt\,\delta(t)$?

Problems

17.1 Calculate the mean, the variance, the skewness, and the kurtosis excess for a Gaussian: $f(g) = Ae^{-B(g-g_0)^2}$ $(-\infty < g < \infty)$. Assume that this function is normalized the same way that Eq. (17.8) is, so that its integral is one.

17.2 Calculate the mean, variance, skewness, and the kurtosis excess for a flat distribution, $f(g) =$ constant, $(0 < g < g_{\text{max}})$. Ans: Var $= g_{\text{m}}^2/12$ kurt. exc. $= -6/5$

17.3 Derive the results stated in Eq. (17.9). Compare $m\bar{v}^2/2$ to $\overline{\text{K.E.}}$ Compare this to the results of problem 2.48.

17.4 Show that you can rewrite Eq. (17.16) as an integral $\int_{-\infty}^{t} dt'\, \frac{1}{m} \cos\omega_0(t-t')F(t')$ and differentiate this directly, showing yet again that (17.15) satisfies the differential equation.

17.5 What are the units of a delta function?

17.6 Show that

$$\delta\big(f(x)\big) = \delta(x - x_0)/|f'(x_0)|$$

where x_0 is the root of f. Assume just one root for now, and the extension to many roots will turn this into a sum as in Eq. (17.28).

17.7 Show that

(a) $x\delta'(x) = -\delta(x)$	**(b)** $x\delta(x) = 0$
(c) $\delta'(-x) = -\delta'(x)$	**(d)** $f(x)\delta(x-a) = f(a)\delta(x-a)$

17.8 Verify that the functions in Eq. (17.20) satisfy the requirements for a delta sequence. Are they normalized to have an integral of one? Sketch each. Sketch Eq. (17.26). It is complex, so sketch both parts. How can a delta sequence be complex? Verify that the imaginary part of this function doesn't contribute.

17.9 What is the analog of Eq. (17.25) if δ_n is a sequence of Gaussians: $\sqrt{n/\pi}\, e^{-nx^2}$?
Ans: $\theta_n(x) = \frac{1}{2}\left[1 + \operatorname{erf}\left(x\sqrt{n}\right)\right]$

17.10 Interpret the functional derivative of the functional in Eq. (17.18): $\delta\,\delta[\phi]/\delta\phi$. Despite appearances, this actually makes sense. Ans: $\delta(x)$

17.11 Repeat the derivation of Eq. (17.33) but with less labor, selecting the form of the function g to simplify the work. In the discussion following this equation, reread the comments on this subject.

17.12 Verify the derivation of Eq. (17.35). Also examine this solution for the cases that x_0 is very large and that it is very small.

17.13 Fill in the steps in section 17.7 leading to the Green's function for $g'' - k^2 g = \delta$.

17.14 Derive the analog of Eq. (17.39) for the case $x < y$.

17.15 Calculate the contribution of the second exponential factor leading to Eq. (17.41).

17.16 Starting with the formulation in Eq. (17.23), what is the result of δ' and of δ'' on a test function? Draw sketches of a typical δ_n, δ'_n, and δ''_n.

17.17 If $\rho(\vec{r}\,) = qa^2\,\partial^2\delta(\vec{r}\,)/\partial z^2$, compute the potential *and* sketch the charge density. You should express your answer in spherical coordinates as well as rectangular, perhaps commenting on the nature of the results and relating it to functions you have encountered before. You can do this calculation in either rectangular or spherical coordinates.
Ans: $(2qa^2/4\pi\epsilon_0)P_2(\cos\theta)/r^3$

17.18 What is a picture of the charge density $\rho(\vec{r}\,) = qa^2\,\partial^2\delta(\vec{r}\,)/\partial x\partial y$?
(Planar quadrupole) What is the potential for this case?

17.19 In Eq. (17.46) I was not at all explicit about which variables are kept constant in each partial derivative. Sort this out for both $\partial/\partial z'$ and for $\partial/\partial u_z$.

17.20 Use the results of the problem 17.16, showing graphs of δ_n and its derivatives. Look again at the statements leading up to Eq. (17.31), that g is continuous, and ask what would happen if it is not. Think of the right hand side of Eq. (17.30) as a δ_n too in this case, and draw a graph of the left side of the same equation if g_n is assumed to change very fast, approaching a discontinuous function as $n\to\infty$. Demonstrate by looking at the graphs of the left and right side of the equation that this *can't* be a solution and so that g must be continuous as claimed.

17.21 Calculate the mean, variance, skewness, and the kurtosis excess for the density $f(g) = A[\delta(g) + \delta(g-g_0) + \delta(g-xg_0)]$. See how these results vary with the parameter x.
Ans: skewness $= 2^{-3/2}(1+x)(x-2)(2x-1)/\big(1-x+x^2\big)$
kurt. excess $= -3 + \frac{3}{4}\big(1+x^4+(1-x)^4\big)/\big(1-x+x^2\big)^2$

17.22 Calculate the potential of a linear quadrupole as in Eq. (17.49). Also, what is the potential of the planar array mentioned there? You should be able to express the first of these in terms of familiar objects.

17.23 (If this seems out of place, it's used in the next problems.) The unit square, $0<x<1$ and $0<y<1$, has area $\int dx\,dy = 1$ over the limits of x and y. Now change the variables to

$$u = \tfrac{1}{2}(x+y) \qquad \text{and} \qquad v = x - y$$

and evaluate the integral, $\int du\,dv$ over the square, showing that you get the same answer. You have only to work out all the limits. Draw a picture. This is a special example of how to change multiple variables of integration. The single variable integral generalizes

$$\text{from} \qquad \int f(x)\,dx = \int f(x)\,\frac{dx}{du}du \qquad \text{to} \qquad \int f(x,y)\,dx\,dy = \int f(x,y)\frac{\partial(x,\,y)}{\partial(u,\,v)}du\,dv$$

where

$$\frac{\partial(x,\,y)}{\partial(u,\,v)} = \det\begin{pmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{pmatrix}$$

For the given change from x, y to u, v show that this Jacobian determinant is one. A discussion of the Jacobian appears in many advanced calculus texts.

17.24 Problem 17.1 asked for the mean and variance for a Gaussian, $f(g) = Ae^{-B(g-g_0)^2}$. Interpreting this as a distribution of grades in a class, what is the resulting distribution of the average of any two students? That is, given this function for all students, what is the resulting distribution of $(g_1+g_2)/2$? What is the mean of this and what is the root-mean-square deviation from the mean? How do these compare to the original distribution? To do this, note that $f(g)dg$ is the fraction of students in the interval g to $g+dg$, so $f(g_1)f(g_2)dg_1\,dg_2$ is the fraction for both. Now make the change of variables

$$x=\frac{1}{2}(g_1+g_2) \qquad \text{and} \qquad y=g_1-g_2$$

then the fraction of these coordinates between x and $x+dx$ and y and $y+dy$ is

$$f(g_1)f(g_2)dx\,dy=f(x+y/2)f(x-y/2)dx\,dy$$

Note where the result of the preceding problem is used here. For fixed x, integrate over all y in order to give you the fraction between x and $x+dx$. That is the distribution function for $(g_1+g_2)/2$. [Complete the square.] Ans: Another Gaussian with the same mean and with rms deviation from the mean decreased by a factor $\sqrt{2}$.

17.25 Same problem as the preceding one, but the initial function is

$$f(g)=\frac{a/\pi}{a^2+g^2} \qquad (-\infty<g<\infty)$$

In this case however, you don't have to evaluate the mean and the rms deviation. Show why not.
Ans: The result reproduces the original $f(g)$ exactly, with no change in the spread. These two problems illustrate examples of "stable distributions," for which the distribution of the average of two variables has the same form as the original distribution, changing at most the widths. There are an infinite number of other stable distributions, but there are precisely three that have simple and explicit forms. These examples show two of them. The Residue Theorem helps here.

17.26 Same problem as the preceding two, but the initial function is

(a) $f(g)=1/g_{\text{max}}$ for $0<g<g_{\text{max}}$ **(b)** $f(g)=\frac{1}{2}\big[\delta(g-g_1)+\delta(g-g_2)\big]$

17.27 In the same way as defined in Eq. (17.10), what is the functional derivative of Eq. (17.5)?

17.28 Rederive Eq. (17.27) by choosing an explicit delta sequence, $\delta_n(x)$.

17.29 Verify the result in Eq. (17.36).