Differential Equations

The subject of ordinary differential equations encompasses such a large field that you can make a profession of it. There are however a small number of techniques in the subject that you *have* to know. These are the ones that come up so often in physical systems that you not need both the skills to use them and the intuition about what they will do. That small group of methods is what I'll concentrate on in this chapter.

4.1 Linear Constant-Coefficient

A differential equation such as

$$\left(\frac{d^2x}{dt^2}\right)^3 + t^2x^4 + 1 = 0$$

relating acceleration to position and time, is not one that I'm especially eager to solve, and one of the things that makes it difficult is that it is non-linear. This means that starting with two solutions $x_1(t)$ and $x_2(t)$, the sum $x_1 + x_2$ is not a solution; look at all the cross-terms you get if you try to plug the sum into the equation and have to cube the sum of the second derivatives. Also if you multiply $x_1(t)$ itself by 2 you no longer have a solution.

An equation such as

$$e^t\frac{d^3x}{dt^3} + t^2\frac{dx}{dt} - x = 0$$

may be a mess to solve, but if you have two solutions, $x_1(t)$ and $x_2(t)$ then the sum $\alpha x_1 + \beta x_2$ is also a solution. Proof? Plug in:

$$\begin{aligned}&e^t\frac{d^3(\alpha x_1+\beta x_2)}{dt^3} + t^2\frac{d(\alpha x_1+\beta x_2)}{dt} - (\alpha x_1+\beta x_2)\\ &\quad = \alpha\left(e^t\frac{d^3x_1}{dt^3} + t^2\frac{dx_1}{dt} - x_1\right) + \beta\left(e^t\frac{d^3x_2}{dt^3} + t^2\frac{dx_2}{dt} - x_2\right) = 0\end{aligned}$$

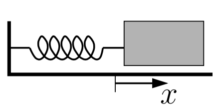
This is called a linear, homogeneous equation because of this property. A similar-looking equation,

$$e^t\frac{d^3x}{dt^3} + t^2\frac{dx}{dt} - x = t$$

does not have this property, though it's close. It is called a linear, inhomogeneous equation. If $x_1(t)$ and $x_2(t)$ are solutions to this, then if I try their sum as a solution I get $2t = t$, and that's no solution, but it misses working only because of the single term on the right, and that will make it not too far removed from the preceding case.

One of the most common sorts of differential equations that you see is an especially simple one to solve. That's part of the reason it's so common. This is the linear, constant-coefficient, differential equation. If you have a mass tied to the end of a spring and the other end of the spring is fixed, the force applied to the mass by the spring is to a good approximation proportional to the distance that the mass has moved from its equilibrium position.

If the coordinate x is measured from the mass's equilibrium position, the equation $\vec F = m\vec a$ says



$$m\frac{d^2x}{dt^2} = -kx \tag{4.1}$$

If there's friction (and there's *always* friction), the force has another term. Now how do you describe friction mathematically? The common model for dry friction is that the magnitude of the force is independent of the magnitude of the mass's velocity and opposite to the direction of the velocity. If you try to write that down in a compact mathematical form you get something like

$$\vec{F}_{\text{friction}} = -\mu_{\text{k}} F_{\text{N}} \frac{\vec{v}}{|\vec{v}|} \tag{4.2}$$

This is hard to work with. It can be done, but I'm going to do something different. (See problem 4.31 however.) Wet friction is easier to handle mathematically because when you lubricate a surface, the friction becomes velocity dependent in a way that is, for low speeds, proportional to the velocity.

$$\vec{F}_{\text{friction}} = -b\vec{v} \tag{4.3}$$

Neither of these two representations is a completely accurate description of the way friction works. That's far more complex than either of these simple models, but these approximations are good enough for many purposes and I'll settle for them.

Assume "wet friction" and the differential equation for the motion of m is

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} \tag{4.4}$$

This is a second order, linear, homogeneous differential equation, which simply means that the highest derivative present is the second, the sum of two solutions is a solution, and a constant multiple of a solution is a solution. That the coefficients are constants makes this an easy equation to solve.

All you have to do is to recall that the derivative of an exponential is an exponential. $de^t/dt = e^t$. Substitute this exponential for $x(t)$, and of course it can't work as a solution; it doesn't even make sense dimensionally. What is e to the power of a day? You need something in the exponent to make it dimensionless, $e^{\alpha t}$. Also, the function x is supposed to give you a position, with dimensions of length. Use another constant: $x(t) = Ae^{\alpha t}$. Plug *this* into the differential equation (4.4) to find

$$mA\alpha^2 e^{\alpha t} + bA\alpha e^{\alpha t} + kAe^{\alpha t} = Ae^{\alpha t}\big[m\alpha^2 + b\alpha + k\big] = 0$$

The product of factors is zero, and the only way that a product of two numbers can be zero is if one of the numbers is zero. The exponential never vanishes, and for a non-trivial solution $A \neq 0$, so all that's left is the polynomial in α.

$$m\alpha^2 + b\alpha + k = 0, \qquad \text{with solutions} \qquad \alpha = \frac{-b \pm \sqrt{b^2 - 4km}}{2m} \tag{4.5}$$

The position function is then

$$x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} \tag{4.6}$$

where A and B are arbitrary constants and α_1 and α_2 are the two roots.

Isn't this supposed to be oscillating? It is a harmonic oscillator after all, but the exponentials don't look very oscillatory. If you have a mass on the end of a spring and the entire system is immersed in honey, it won't do much oscillating! Translated into mathematics, this says that if the constant b is too large, there is no oscillation. In the equation for α, if b is large enough the argument of the square root is positive, and both α's are real — no oscillation. Only if b is small enough does the argument of the square root become negative; then you get complex values for the α's and hence oscillations.

Push this to the extreme case where the damping vanishes: $b = 0$. Then $\alpha_1 = i\sqrt{k/m}$ and $\alpha_2 = -i\sqrt{k/m}$. Denote $\omega_0 = \sqrt{k/m}$.

$$x(t) = Ae^{i\omega_0 t} + Be^{-i\omega_0 t} \tag{4.7}$$

You can write this in other forms using sines and cosines, see problem 4.10. To determine the arbitrary constant A and B you need two equations. They come from some additional information about the problem, typically some initial conditions. Take a specific example in which you start from the origin with a kick, $x(0) = 0$ and $\dot{x}(0) = v_0$.

$$x(0) = 0 = A + B, \qquad \dot{x}(0) = v_0 = i\omega_0 A - i\omega_0 B$$

Solve for A and B to get $A = -B = v_0/(2i\omega_0)$. Then

$$x(t) = \frac{v_0}{2i\omega_0}\left[e^{i\omega_0 t} - e^{-i\omega_0 t}\right] = \frac{v_0}{\omega_0}\sin\omega_0 t$$

As a check on the algebra, use the first term in the power series expansion of the sine function to see how x behaves for small t. The sine factor is $\sin\omega_0 t \approx \omega_0 t$, and then $x(t)$ is approximately $v_0 t$, just as it should be. Also notice that despite all the complex numbers, the final answer is real. This is another check on the algebra.

Damped Oscillator

If there is damping, but not too much, then the α's have an imaginary part *and* a negative real part. (Is it important whether it's negative or not?)

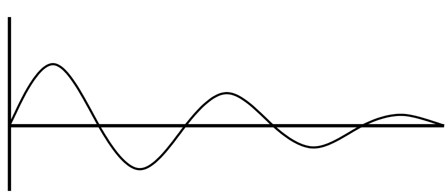
$$\alpha = \frac{-b \pm i\sqrt{4km - b^2}}{2m} = -\frac{b}{2m} \pm i\omega', \qquad \text{where} \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \tag{4.8}$$

This represents a damped oscillation and has frequency a bit lower than the one in the undamped case. Use the same initial conditions as above and you will get similar results (let $\gamma = b/2m$)

$$\begin{aligned} x(t) &= Ae^{(-\gamma + i\omega')t} + Be^{(-\gamma - i\omega')t} \\ x(0) = A + B &= 0, \qquad v_x(0) = (-\gamma + i\omega')A + (-\gamma - i\omega')B = v_0 \end{aligned} \tag{4.9}$$

The two equations for the unknowns A and B imply $B = -A$ and

$$2i\omega' A = v_0, \qquad \text{so} \qquad x(t) = \frac{v_0}{2i\omega'}e^{-\gamma t}\left[e^{i\omega' t} - e^{-i\omega' t}\right] = \frac{v_0}{\omega'}e^{-\gamma t}\sin\omega' t \tag{4.10}$$



For small values of t, the first terms in the power series expansion of this result are

$$x(t) = \frac{v_0}{\omega'}[1 - \gamma t + \gamma^2 t^2/2 - \ldots][\omega' t - \omega'^3 t^3/6 + \ldots] = v_0 t - v_0\gamma t^2 + \ldots$$

The first term is what you should expect, as the initial velocity is $v_x = v_0$. The negative sign in the next term says that it doesn't move as far as it would without the damping, but analyze it further.

Does it have the right size as well as the right sign? It is $-v_0\gamma t^2 = -v_0(b/2m)t^2$. But that's an acceleration: $a_x t^2/2$. It says that the acceleration just after the motion starts is $a_x = -bv_0/m$. Is that what you should expect? As the motion starts, the mass hasn't gone very far so the spring doesn't yet exert much force. The viscous friction is however $-bv_x$. Set that equal to ma_x and you see that $-v_0\gamma t^2$ has precisely the right value:

$$x(t) \approx v_0 t - v_0\gamma t^2 = v_0 t - v_0\frac{b}{2m}t^2 = v_0 t + \frac{1}{2}\frac{-bv_0}{m}t^2$$

The last term says that the acceleration starts as $a_x = -bv_0/m$, as required.

In Eq. (4.8) I assumed that the two roots of the quadratic, the two α's, are different. What if they aren't? Then you have just one value of α to use in defining the solution $e^{\alpha t}$ in Eq. (4.9). You now have just one arbitrary constant with which to match two initial conditions. You're stuck. See problem 4.11 to understand how to handle this case (critical damping). It's really a special case of what I've already done.

What is the energy for this damped oscillator? The kinetic energy is $mv^2/2$ and the potential energy for the spring is $kx^2/2$. Is the sum constant? No.

$$\text{If} \quad F_x = ma_x = -kx + F_{x,\text{frict}}, \quad \text{then}$$
$$\frac{dE}{dt} = \frac{d}{dt}\frac{1}{2}\big(mv^2 + kx^2\big) = mv\frac{dv}{dt} + kx\frac{dx}{dt} = v_x\big(ma_x + kx\big) = F_{x,\text{frict}}v_x \tag{4.11}$$

"Force times velocity" is a common expression for power, and this says that the total energy is decreasing according to this formula. For the wet friction used here, this is $dE/dt = -bv_x^2$, and the energy decreases exponentially on average.

4.2 Forced Oscillations

What happens if the equation is inhomogeneous? That is, what if there is a term that doesn't involve x or its derivatives at all. In this harmonic oscillator example, apply an extra external force. Maybe it's a constant; maybe it's an oscillating force; it can be anything you want not involving x.

$$m\frac{d^2x}{dt^2} = -kx - b\frac{dx}{dt} + F_{\text{ext}}(t) \tag{4.12}$$

The key result that you need for this class of equations is very simple to state and not too difficult to implement. It is a procedure for attacking any linear inhomogeneous differential equation and consists of three steps.

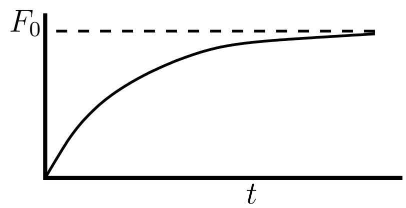
1. Temporarily throw out the inhomogeneous term [here $F_{\text{ext}}(t)$] and completely solve the resulting homogeneous equation. In the current case that's what you just saw when I worked out the solution to the differential equation $md^2x/dt^2 + bdx/dt + kx = 0$. [$x_{\text{hom}}(t)$]
2. Find any *one* solution to the full inhomogeneous equation. Note that for step one you have to have all the arbitrary constants present; for step two you do not. [$x_{\text{inh}}(t)$]
3. Add the results of steps one and two. [$x_{\text{hom}}(t) + x_{\text{inh}}(t)$]

I've already done step one. To carry out the next step I'll start with a particular case of the forcing function. If $F_{\text{ext}}(t)$ is simple enough, you should be able to *guess* the answer to step two. If it's a constant, then a constant will work for x. If it's a sine or cosine, then you can guess that a sine or cosine or a combination of the two should work. If it's an exponential, then guess an exponential — remember that the derivative of an exponential is an exponential. If it's the sum of two terms, such as a constant and an exponential, it's easy to verify that you add the results that you get for the two cases separately. If the forcing function is too complicated for you to guess a solution then there's a general method using Green's functions that I'll get to in section 4.6.

Choose a specific example

$$F_{\text{ext}}(t) = F_0\big[1 - e^{-\beta t}\big] \tag{4.13}$$

This starts at zero and builds up to a final value of F_0. It does it slowly or quickly depending on β.



Start with the first term, F_0, for external force in Eq. (4.12). Try $x(t) = C$ and plug into that equation to find

$$kC = F_0$$

This is simple and determines C.

Next, use the second term as the forcing function, $-F_0 e^{-\beta t}$. Guess a solution $x(t) = C' e^{-\beta t}$ and plug in. The exponential cancels, leaving

$$mC'\beta^2 - bC'\beta + kC' = -F_0 \qquad \text{or} \qquad C' = \frac{-F_0}{m\beta^2 - b\beta + k}$$

The total solution for the inhomogeneous part of the equation is then the sum of these two expressions.

$$x_{\text{inh}}(t) = F_0\left(\frac{1}{k} - \frac{1}{m\beta^2 - b\beta + k}e^{-\beta t}\right)$$

The homogeneous part of Eq. (4.12) has the solution found in Eq. (4.6) and the total is

$$x(t) = x_{\text{hom}}(t) + x_{\text{inh}}(t) = x(t) = Ae^{\alpha_1 t} + Be^{\alpha_2 t} + F_0\left(\frac{1}{k} - \frac{1}{m\beta^2 - b\beta + k}e^{-\beta t}\right) \tag{4.14}$$

There are two arbitrary constants here, and this is what you need because you have to be able to specify the initial position and the initial velocity independently; this is a second order differential equation after all. Take for example the conditions that the initial position is zero and the initial velocity is zero. Everything is at rest until you start applying the external force. This provides two equations for the two unknowns.

$$x(0) = 0 = A + B + F_0\frac{m\beta^2 - b\beta}{k(m\beta^2 - b\beta + k)}$$
$$\dot{x}(0) = 0 = A\alpha_1 + B\alpha_2 + F_0\frac{\beta}{m\beta^2 - b\beta + k}$$

Now all you have to do is solve the two equations in the two unknowns A and B. Take the first, multiply it by α_2 and subtract the second. This gives A. Do the same with α_1 instead of α_2 to get B. The results are

$$A = \frac{1}{\alpha_1 - \alpha_2}F_0\frac{\alpha_2(m\beta^2 - b\beta) - k\beta}{k(m\beta^2 - b\beta + k)}$$

Interchange α_1 and α_2 to get B.

The final result is

$$x(t) = \frac{F_0}{\alpha_1-\alpha_2}\frac{\big(\alpha_2(m\beta^2-b\beta)-k\beta\big)e^{\alpha_1 t}-\big(\alpha_1(m\beta^2-b\beta)-k\beta\big)e^{\alpha_2 t}}{k(m\beta^2-b\beta+k)} + F_0\left(\frac{1}{k}-\frac{1}{m\beta^2-b\beta+k}e^{-\beta t}\right) \tag{4.15}$$

If you think this is messy and complicated, you haven't seen messy and complicated. When it takes 20 pages to write out the equation, then you're entitled say that it is starting to become involved.

Why not start with a simpler example, one without all the terms? The reason is that a complex expression is often easier to analyze than a simple one. There are more things that you can do to it, and so more opportunities for it to go wrong. The problem isn't finished until you've analyzed the supposed solution. After all, I may have made some errors in algebra along the way. Also, analyzing the solution is the way you learn how these functions *work.*

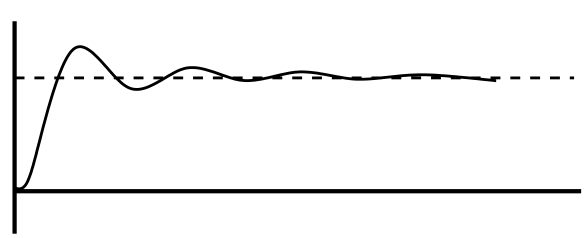
1. Everything in the solution is proportional to F_0 and that's not surprising.
2. I'll leave it as an exercise to check the dimensions.
3. A key parameter to vary is β. What should happen if it is either very large or very small? In the former case the exponential function in the force drops to zero quickly so the force jumps from zero to F_0 in a very short time — a step in the limit that $\beta\to 0$.
4. If β is very small the force turns on very gradually and gently, as though you are being very careful not to disturb the system.

Take point 3 above: for large β the dominant terms in both numerator and denominator everywhere are the $m\beta^2$ terms. This result is then very nearly

$$\begin{aligned} x(t) &\approx \frac{F_0}{\alpha_1-\alpha_2}\frac{\big(\alpha_2(m\beta^2)\big)e^{\alpha_1 t}-\big(\alpha_1(m\beta^2)\big)e^{\alpha_2 t}}{km\beta^2} + F_0\left(\frac{1}{k}-\frac{1}{(m\beta^2)}e^{-\beta t}\right) \\ &\approx \frac{F_0}{k(\alpha_1-\alpha_2)}\big[(\alpha_2 e^{\alpha_1 t}-\alpha_1 e^{\alpha_2 t}\big] + F_0\frac{1}{k} \end{aligned}$$

Use the notation of Eq. (4.9) and you have

$$\begin{aligned} x(t) &\approx \frac{F_0}{k\big(-\gamma+i\omega'-(-\gamma-i\omega')\big)}\big[((-\gamma-i\omega')e^{(-\gamma+i\omega')t}-(-\gamma+i\omega')e^{(-\gamma-i\omega')t}\big] + F_0\frac{1}{k} \\ &= \frac{F_0 e^{-\gamma t}}{k(2i\omega')}\big[-2i\gamma\sin\omega' t-2i\omega'\cos\omega' t\big] + F_0\frac{1}{k} \\ &= \frac{F_0 e^{-\gamma t}}{k}\big[-\frac{\gamma}{\omega'}\sin\omega' t-\cos\omega' t\big] + F_0\frac{1}{k} \end{aligned} \tag{4.16}$$



At time $t = 0$ this is still zero even with the approximations. That's comforting, but if it hadn't happened it's not an insurmountable disaster. This is an approximation to the exact answer after all, so it could happen that the initial conditions are obeyed only approximately. The exponential terms have oscillations and damping, so the mass oscillates about its eventual equilibrium position and after a long enough time the oscillations die out and you are left with the equilibrium solution $x = F_0/k$.

Look at point 4 above: For small β the β^2 terms in Eq. (4.15) are small compared to the β terms to which they are added or subtracted. The numerators of the terms with $e^{\alpha t}$ are then proportional to β. The denominator of the same terms has a $k - b\beta$ in it. That means that as $\beta \to 0$, the numerator of the homogeneous term approaches zero and its denominator doesn't. The last terms, that came from the inhomogeneous part, don't have any β in the numerator so they don't vanish in this limit. The approximate final result then comes solely from the $x_{\text{inh}}(t)$ term.

$$x(t) \approx F_0 \frac{1}{k}\left(1 - e^{-\beta t}\right)$$

It doesn't oscillate at all and just gradually moves from equilibrium to equilibrium as time goes on. It's what you get if you go back to the differential equation (4.12) and say that the acceleration and the velocity are negligible.

$$m\frac{d^2x}{dt^2}[\approx 0] = -kx - b\frac{dx}{dt}[\approx 0] + F_{\text{ext}}(t) \qquad \Longrightarrow \qquad x \approx \frac{1}{k}F_{\text{ext}}(t)$$

The spring force nearly balances the external force at all times; this is "quasi-static," in which the external force is turned on so slowly that it doesn't cause any oscillations.

4.3 Series Solutions

A linear, second order differential equation can always be rearranged into the form

$$y'' + P(x)y' + Q(x)y = R(x) \tag{4.17}$$

If at some point x_0 the functions P and Q are well-behaved, if they have convergent power series expansions about x_0, then this point is called a "*regular point*" and you can expect good behavior of the solutions there — at least if R is also regular there.

I'll look just at the case for which the inhomogeneous term $R = 0$. If P or Q has a singularity at x_0, perhaps something such as $1/(x - x_0)$ or $\sqrt{x - x_0}$, then x_0 is called a "*singular point*" of the differential equation.

Regular Singular Points

The most important special case of a singular point is the "*regular singular point*" for which the behaviors of P and Q are not too bad. Specifically this requires that $(x - x_0)P(x)$ and $(x - x_0)^2Q(x)$ have no singularity at x_0. For example

$$y'' + \frac{1}{x}y' + \frac{1}{x^2}y = 0 \qquad \text{and} \qquad y'' + \frac{1}{x^2}y' + xy = 0$$

have singular points at $x = 0$, but the first one is a regular singular point and the second one is not. The importance of a regular singular point is that there is a procedure guaranteed to find a solution near a regular singular point (Frobenius series). For the more general singular point there is no guaranteed procedure (though there are a few tricks* that sometimes work).

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* The book by Bender and Orszag: "Advanced mathematical methods for scientists and engineers" is a very readable source for this and many other topics.

Examples of equations that show up in physics problems are

$$\begin{aligned} y'' + y &= 0 \\ (1-x^2)y'' - 2xy' + \ell(\ell+1)y &= 0 \qquad \text{regular singular points at } \pm 1 \\ x^2y'' + xy' + (x^2 - n^2)y &= 0 \qquad \text{regular singular point at zero} \\ xy'' + (\alpha + 1 - x)y' + ny &= 0 \qquad \text{regular singular point at zero} \end{aligned} \tag{4.18}$$

These are respectively the classical simple harmonic oscillator, Legendre equation, Bessel equation, generalized Laguerre equation.

A standard procedure to solve these equations is to use series solutions, but not just the standard power series such as those in Eq. (2.4). Essentially, you assume that there is a solution in the form of an infinite series and you systematically compute the terms of the series. I'll pick the Bessel equation from the above examples, as the other three equations are done the same way. The parameter n in that equation is often an integer, but it can be anything. It's common for it to be $1/2$ or $3/2$ or sometimes even imaginary, but there's no need to make any assumptions about it for now.

Assume a solution in the form :

$$\text{Frobenius Series:} \qquad y(x) = \sum_0^\infty a_k x^{k+s} \qquad (a_0 \neq 0) \tag{4.19}$$

If $s = 0$ or a positive integer, this is just the standard Taylor series you saw so much of in chapter two, but this simple-looking extension makes it much more flexible and suited for differential equations. It often happens that s is a fraction or negative, but this case is no harder to handle than the Taylor series. For example, what is the series expansion of $(\cos x)/x$ about the origin? This is singular at zero, but it's easy to write the answer anyway because you already know the series for the cosine.

$$\frac{\cos x}{x} = \frac{1}{x} - \frac{x}{2} + \frac{x^3}{24} - \frac{x^5}{720} + \cdots$$

It starts with the term $1/x$ corresponding to $s = -1$ in the Frobenius series.

Always assume that $a_0 \neq 0$, because that just defines the coefficient of the most negative power, x^s. If you allow it be zero, that's just the same as redefining s and it gains nothing except confusion. Plug this into the Bessel differential equation.

$$\begin{aligned} x^2y'' + xy' + (x^2 - n^2)y &= 0 \\ x^2 \sum_{k=0}^\infty a_k(k+s)(k+s-1)x^{k+s-2} + x\sum_{k=0}^\infty a_k(k+s)x^{k+s-1} + (x^2 - n^2)\sum_{k=0}^\infty a_k x^{k+s} &= 0 \\ \sum_{k=0}^\infty a_k(k+s)(k+s-1)x^{k+s} + \sum_{k=0}^\infty a_k(k+s)x^{k+s} + \sum_{k=0}^\infty a_k x^{k+s+2} - n^2\sum_0^\infty a_k x^{k+s} &= 0 \\ \sum_{k=0}^\infty a_k\big[(k+s)(k+s-1) + (k+s) - n^2\big]x^{k+s} + \sum_{k=0}^\infty a_k x^{k+s+2} &= 0 \end{aligned}$$

The coefficients of all the like powers of x must match, and in order to work out the matches efficiently, and so as not to get myself confused in a mess of indices, I'll make an explicit change of the index in the sums. *Do this trick every time. It keeps you out of trouble.*

Let $\ell = k$ in the first sum. Let $\ell = k + 2$ in the second. *Explicitly* show the limits of the index on the sums, or you're bound to get it wrong.

$$\sum_{\ell=0}^{\infty} a_\ell \big[(\ell+s)^2 - n^2\big] x^{\ell+s} + \sum_{\ell=2}^{\infty} a_{\ell-2} x^{\ell+s} = 0$$

The lowest power of x in this equation comes from the $\ell = 0$ term in the first sum. That coefficient of x^s must vanish. $(a_0 \neq 0)$

$$a_0\big[s^2 - n^2\big] = 0 \tag{4.20}$$

This is called the *indicial* equation. It determines s, or in this case, maybe two s's. After this, set to zero the coefficient of $x^{\ell+s}$.

$$a_\ell\big[(\ell+s)^2 - n^2\big] + a_{\ell-2} = 0 \tag{4.21}$$

This determines a_2 in terms of a_0; it determines a_4 in terms of a_2 *etc.*

$$a_\ell = -a_{\ell-2}\frac{1}{(\ell+s)^2 - n^2}, \qquad \ell = 2,\ 4,\ \ldots$$

For example, if $n = 0$, the indicial equation says $s = 0$.

$$a_2 = -a_0\frac{1}{2^2}, \qquad a_4 = -a_2\frac{1}{4^2} = +a_0\frac{1}{2^2 4^2}, \qquad a_6 = -a_4\frac{1}{6^2} = -a_0\frac{1}{2^2 4^2 6^2}$$

$$a_{2k} = (-1)^k a_0\frac{1}{2^{2k}k!^2} \qquad \text{then} \qquad y(x) = a_0\sum_{k=0}^{\infty}(-1)^k\frac{(x/2)^{2k}}{(k!)^2} = a_0 J_0(x) \tag{4.22}$$

and in the last equation I rearranged the factors and used the standard notation for the Bessel function, $J_n(x)$.

This is a second order differential equation. What about the other solution? This Frobenius series method is guaranteed to find one solution near a regular singular point. Sometimes it gives both but not always, and in this example it produces only one. There are procedures that will let you find the second solution to this sort of second order differential equation. See problem 4.49 for one such method.

For the case $n = 1/2$ the calculations just above will produce two solutions. The indicial equation gives $s = \pm 1/2$. After that, the recursion relation for the coefficients give

$$a_\ell = -a_{\ell-2}\frac{1}{(\ell+s)^2 - n^2} = -a_{\ell-2}\frac{1}{\ell^2 + 2\ell s} = -a_{\ell-2}\frac{1}{\ell(\ell+2s)} = -a_{\ell-2}\frac{1}{\ell(\ell\pm 1)}$$

For the $s = +1/2$ result

$$a_2 = -a_0\frac{1}{2\cdot 3} \qquad\qquad a_4 = -a_2\frac{1}{4\cdot 5} = +a_0\frac{1}{2\cdot 3\cdot 4\cdot 5}$$

$$a_{2k} = (-1)^k a_0\frac{1}{(2k+1)!}$$

This solution is then

$$y(x) = a_0 x^{1/2}\left[1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots\right]$$

This series looks suspiciously like the series for the sine function, but is has some of the x's or some of the factorials in the wrong place. You can fix that if you multiply the series in brackets by x. You then have

$$y(x) = a_0 x^{-1/2}\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right] = a_0 \frac{\sin x}{x^{1/2}} \tag{4.23}$$

I'll leave it to problem 4.15 for you to find the other solution.

Do you need to use a Frobenius series instead of just a power series for all differential equations? No, but I recommend it. If you are expanding about a regular point of the equation then a power series will work, but I find it more systematic to use the same method for all cases. It's less prone to error.

4.4 Some General Methods

It is important to be familiar with the arsenal of special methods that work on special types of differential equations. What if you encounter an equation that doesn't fit these special methods? There are some techniques that you should be familiar with, even if they are mostly not ones that you will want to use often. Here are a couple of methods that can get you started, and there's a much broader set of approaches under the heading of numerical analysis; you can explore those in section 11.5.

If you have a first order differential equation, $dx/dt = f(x,t)$, with initial condition $x(t_0) = x_0$ then you can follow the spirit of the series method, computing successive orders in the expansion. Assume for now that the function f is smooth, with as many derivatives as you want, then use the chain rule a lot to get the higher derivatives of x

$$\begin{aligned}
\frac{dx}{dt} &= f(x,t) \\
\frac{d^2x}{dt^2} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} = \ddot{x} = f_t + f_x\dot{x} \\
\dddot{x} &= f_{tt} + 2f_{xt}\dot{x} + f_{xx}\dot{x}^2 + f_x\ddot{x} = f_{tt} + 2f_{xt}\dot{x} + f_{xx}\dot{x}^2 + f_x[f_t + f_x\dot{x}] \\
x(t) &= x_0 + f(x_0,t_0)(t-t_0) + \tfrac{1}{2}\ddot{x}(t_0)(t-t_0)^2 + \tfrac{1}{6}\dddot{x}(t_0)x(t_0)(t-t_0)^3 + \cdots
\end{aligned} \tag{4.24}$$

Here the dot-notation ($\dot{x}$ *etc.*) is a standard shorthand for derivative with respect to time. This is unlike using a prime for derivative, which is with respect to anything you want. These equations show that once you have the initial data (t_0, x_0), you can compute the next derivatives from them and from the properties of f. Of course if f is complicated this will quickly become a mess, but even then it can be useful to compute the first few terms in the power series expansion of x.

For example, $\dot{x} = f(x,t) = Ax^2(1+\omega t)$ with $t_0 = 0$ and $x_0 = \alpha$.

$$\dot{x}_0 = A\alpha^2, \qquad \ddot{x}_0 = A\alpha^2\omega + 2A^2\alpha^3, \qquad \dddot{x}_0 = 4A^2\alpha^3\omega + 2A^3\alpha^4 + 2A\alpha\big[A\alpha^2\omega + 2A^2\alpha^3\big] \tag{4.25}$$

If $A = 1/\,\text{m}\cdot\text{s}$ and $\omega = 1/\text{s}$ with $\alpha = 1\,\text{m}$ this is

$$x(t) = 1 + t + \tfrac{3}{2}t^2 + 2t^3 + \cdots$$

You can also solve this example exactly and compare the results to check the method.

What if you have a second order differential equation? Pretty much the same thing, though it is sometimes convenient to make a slight change in the appearance of the equations when you do this.

$$\ddot{x} = f(x,\dot{x},t) \qquad \text{can be written} \qquad \dot{x} = v, \qquad \dot{v} = f(x,v,t) \tag{4.26}$$

so that it looks like two simultaneous first order equations. Either form will let you compute the higher derivatives, but the second one often makes for a less cumbersome notation. You start by knowing t_0, x_0, and now $v_0 = \dot{x}_0$.

Some of the numerical methods you will find in chapter 11 start from the ideas of these expansions, but then develop them along different lines.

There is an iterative methods that of more theoretical than practical importance, but it's easy to understand. I'll write it for a first order equation, but you can rewrite it for the second (or higher) order case by doing the same thing as in Eq. (4.26).

$$\dot{x} = f(x,t) \qquad \text{with} \qquad x(t_0) = x_0 \qquad \text{generates} \qquad x_1(t) = \int_{t_0}^{t} dt'\, f(x_0, t')$$

This is not a solution of the differential equation, but it forms the starting point to find one because you can iterate this approximate solution x_1 to form an improved approximation.

$$x_k(t) = \int_{t_0}^{t} dt'\, f\big(x_{k-1}(t'), t'\big), \quad k = 2, 3, \ldots \tag{4.27}$$

This will form a sequence that is usually different from that of the power series approach, though the end result better be the same. This iterative approach is used in one proof that shows under just what circumstances this differential equation $\dot{x} = f$ has a unique solution.

4.5 Trigonometry via ODE's

The differential equation $u'' = -u$ has two independent solutions. The point of this exercise is to derive all (or at least some) of the standard relationships for sines and cosines *strictly from the differential equation.* The reasons for spending some time on this are twofold. First, it's neat. Second, you have to get used to manipulating a differential equation in order to find properties of its solutions. This is essential in the study of Fourier series as you will see in section 5.3.

Two solutions can be defined when you specify boundary conditions. Call the functions $c(x)$ and $s(x)$, and specify their respective boundary conditions to be

$$c(0) = 1, \quad c'(0) = 0, \qquad \text{and} \qquad s(0) = 0, \quad s'(0) = 1 \tag{4.28}$$

What is $s'(x)$? First observe that s' satisfies the same differential equation as s and c:

$$u'' = -u \implies (u')'' = (u'')' = -u', \qquad \text{and that shows the desired result.}$$

This in turn implies that s' is a linear combination of s and c, as that is the most general solution to the original differential equation.

$$s'(x) = Ac(x) + Bs(x)$$

Use the boundary conditions:

$$s'(0) = 1 = Ac(0) + Bs(0) = A$$

From the differential equation you also have

$$s''(0) = -s(0) = 0 = Ac'(0) + Bs'(0) = B$$

Put these together and you have

$$s'(x) = c(x) \qquad \text{And a similar calculation shows} \qquad c'(x) = -s(x) \tag{4.29}$$

What is $c(x)^2 + s(x)^2$? Differentiate this expression to get

$$\frac{d}{dx}[c(x)^2 + s(x)^2] = 2c(x)c'(x) + 2s(x)s'(x) = -2c(x)s(x) + 2s(x)c(x) = 0$$

This combination is therefore a constant. What constant? Just evaluate it at $x = 0$ and you see that the constant is one. There are many more such results that you can derive, but that's left for the exercises, problem 4.21 *et seq.*

4.6 Green's Functions

Is there a general way to find the solution to the whole harmonic oscillator inhomogeneous differential equation? One that does not require guessing the form of the solution and applying initial conditions? Yes there is. It's called the method of Green's functions. The idea behind it is that you can think of any force as a sequence of short, small kicks. In fact, because of the atomic nature of matter, that's not so far from the truth. If you can figure out the result of an impact by one molecule, you can add the results of many such kicks to get the answer for 10^{23} molecules.

I'll start with the simpler case where there's no damping, $b = 0$ in the harmonic oscillator equation.

$$m\ddot{x} + kx = F_{\text{ext}}(t) \tag{4.30}$$

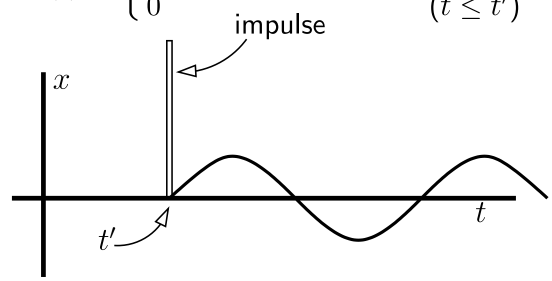
Suppose that everything is at rest at the origin and then at time t' the external force provides a small impulse. The motion from that point on will be a sine function starting at t',

$$A\sin\big(\omega_0(t-t')\big) \qquad (t > t') \tag{4.31}$$

The amplitude will depend on the strength of the kick. A constant force F applied for a very short time, $\Delta t'$, will change the momentum of the mass by $m\Delta v_x = F\Delta t'$. If this time interval is short enough the mass doesn't have a chance to move very far before the force is turned off, then from that time on it's subject only to the $-kx$ force. This kick gives m a velocity $F\Delta t'/m$, and that's what determines the unknown constant A.

Just after $t = t'$, $v_x = A\omega_0 = F\Delta t'/m$. This determines A, so the position of m is

$$x(t) = \begin{cases} \frac{F\Delta t'}{m\omega_0}\sin\big(\omega_0(t-t')\big) & (t > t') \\ 0 & (t \le t') \end{cases} \tag{4.32}$$

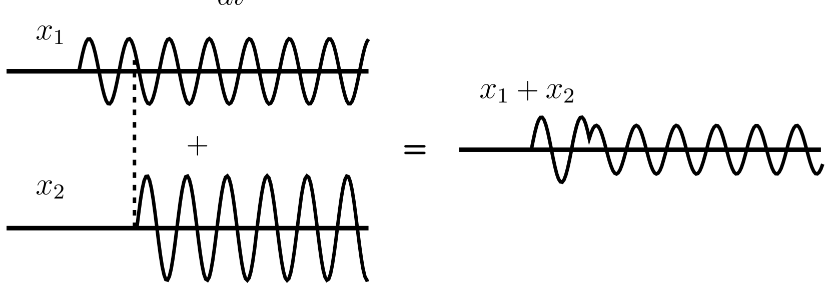


When the external force is the sum of two terms, the total solution is the sum of the solutions for the individual forces. If an impulse at one time gives a solution Eq. (4.32), an impulse at a later time gives a solution that starts its motion at that later time. The key fact about the equation that you're trying to solve is that it is linear, so you can get the solution for two impulses simply by adding the two simpler solutions.

$$m\frac{d^2x_1}{dt^2} + kx_1 = F_1(t) \qquad \text{and} \qquad m\frac{d^2x_2}{dt^2} + kx_2 = F_2(t)$$

then

$$m\frac{d^2(x_1+x_2)}{dt^2} + k(x_1+x_2) = F_1(t) + F_2(t)$$



The way to make use of this picture is to take a sequence of contiguous steps. One step follows immediately after the preceding one. If two such impulses are two steps

$$F_0 = \begin{cases} F(t_0) & (t_0 < t < t_1) \\ 0 & \text{(elsewhere)} \end{cases} \qquad \text{and} \qquad F_1 = \begin{cases} F(t_1) & (t_1 < t < t_2) \\ 0 & \text{(elsewhere)} \end{cases}$$

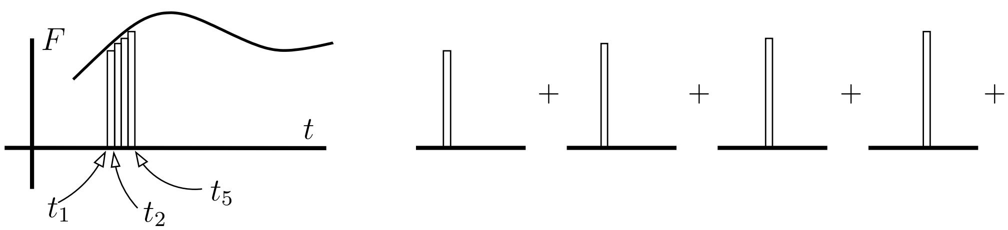
$$m\ddot{x} + kx = F_0 + F_1 \tag{4.33}$$

then if x_0 is the solution to Eq. (4.30) with only the F_0 on its right, and x_1 is the solution with only F_1, then the full solution to Eq. (4.33) is the sum, $x_0 + x_1$.

Think of a general forcing function $F_{x,\text{ext}}(t)$ in the way that you would set up an integral. Approximate it as a sequence of very short steps as in the picture. Between t_k and t_{k+1} the force is essentially $F(t_k)$. The response of m to this piece of the total force is then Eq. (4.32).

$$x_k(t) = \begin{cases} \frac{F(t_k)\Delta t_k}{m\omega_0} \sin\big(\omega_0(t - t_k)\big) & (t > t_k) \\ 0 & (t \le t_k) \end{cases}$$

where $\Delta t_k = t_{k+1} - t_k$.



To complete this idea, the external force is the sum of a lot of terms, the force between t_1 and t_2, that between t_2 and t_3 etc. The total response is the sum of all these individual responses.

$$x(t) = \sum_k \begin{cases} \frac{F(t_k)\Delta t_k}{m\omega_0} \sin\big(\omega_0(t - t_k)\big) & (t > t_k) \\ 0 & (t \le t_k) \end{cases}$$

For a specified time t, only the times t_k before and up to t contribute to this sum. The impulses occurring at the times after the time t can't change the value of $x(t)$; they haven't happened yet. In the limit that $\Delta t_k \to 0$, this sum becomes an integral.

$$x(t) = \int_{-\infty}^{t} dt' \, \frac{F(t')}{m\omega_0} \sin\big(\omega_0(t - t')\big) \tag{4.34}$$

Apply this to an example. The simplest is to start at rest and begin applying a constant force from time zero on.

$$F_{\text{ext}}(t) = \begin{cases} F_0 & (t > 0) \\ 0 & (t \le 0) \end{cases} \qquad\qquad x(t) = \int_0^t dt' \, \frac{F_0}{m\omega_0} \sin\big(\omega_0(t - t')\big)$$

and the last expression applies only for $t > 0$. It is

$$x(t) = \frac{F_0}{m\omega_0^2}\big[1 - \cos(\omega_0 t)\big] \tag{4.35}$$

As a check for the plausibility of this result, look at the special case of small times. Use the power series expansion of the cosine, keeping a couple of terms, to get

$$x(t) \approx \frac{F_0}{m\omega_0^2}\big[1 - \big(1 - (\omega_0 t)^2/2\big)\big] = \frac{F_0}{m\omega_0^2}\frac{\omega_0^2 t^2}{2} = \frac{F_0}{m}\frac{t^2}{2}$$

and this is just the result you'd get for constant acceleration F_0/m. In this short time, the position hasn't changed much from zero, so the spring hasn't had a chance to stretch very far, so it can't apply much force, and you have nearly constant acceleration.

This is a sufficiently important subject that it will be repeated elsewhere in this text. A completely different approach to Green's functions will appear is in section 15.5, and chapter 17 is largely devoted to the subject.

4.7 Separation of Variables

If you have a first order differential equation — I'll be more specific for an example, in terms of x and t — and if you are able to move the variables around until everything involving x and dx is on one side of the equation and everything involving t and dt is on the other side, then you have "separated variables." Now all you have to do is integrate.

For example, the total energy in the undamped harmonic oscillator is $E = mv^2/2 + kx^2/2$. Solve for dx/dt and

$$\frac{dx}{dt} = \sqrt{\frac{2}{m}\big(E - kx^2/2\big)} \tag{4.36}$$

To separate variables, multiply by dt and divide by the right-hand side.

$$\frac{dx}{\sqrt{\frac{2}{m}\big(E - kx^2/2\big)}} = dt$$

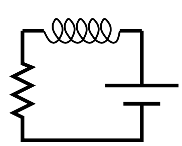
Now it's just manipulation to put this into a convenient form to integrate.

$$\sqrt{\frac{m}{k}}\frac{dx}{\sqrt{(2E/k) - x^2}} = dt, \qquad \text{or} \qquad \int \frac{dx}{\sqrt{(2E/k) - x^2}} = \int \sqrt{\frac{k}{m}}dt$$

Make the substitution $x = a\sin\theta$ and you see that if $a^2 = 2E/k$ then the integral on the left simplifies.

$$\int \frac{a\cos\theta\, d\theta}{a\sqrt{1 - \sin^2\theta}} = \int \sqrt{\frac{k}{m}}dt \qquad \text{so} \qquad \theta = \sin^{-1}\frac{x}{a} = \omega_0 t + C$$
$$\text{or} \qquad x(t) = a\sin(\omega_0 t + C) \qquad \text{where} \qquad \omega_0 = \sqrt{k/m}$$

An electric circuit with an inductor, a resistor, and a battery has a differential equation for the current flow:



$$L\frac{dI}{dt} + IR = V_0 \tag{4.37}$$

Manipulate this into

$$L\frac{dI}{dt} = V_0 - IR, \qquad \text{then} \qquad L\frac{dI}{V_0 - IR} = dt$$

Now integrate this to get

$$L\int \frac{dI}{V_0 - IR} = t + C, \qquad \text{or} \qquad -\frac{L}{R}\ln(V_0 - IR) = t + C$$

Solve for the current I to get

$$RI(t) = V_0 - e^{-(L/R)(t+C)} \tag{4.38}$$

Now does this make sense? Look at the dimensions and you see that it *doesn't*, at least not yet. The problem is the logarithm on the preceding line where you see that its units don't make sense either. How can this be? The differential equation that you started with is correct, so how did the units get messed up? It goes back the the standard equation for integration,

$$\int dx/x = \ln x + C$$

If x is a length for example, then the left side is dimensionless, but this right side is the logarithm of a length. It's a peculiarity of the logarithm that leads to this anomaly. You can write the constant of integration as $C = -\ln C'$ where C' is another arbitrary constant, then

$$\int dx/x = \ln x + C = \ln x - \ln C' = \ln \frac{x}{C'}$$

If C' is a length this is perfectly sensible dimensionally. To see that the dimensions in Eq. (4.38) will work themselves out (this time), put on some initial conditions. Set $I(0) = 0$ so that the circuit starts with zero current.

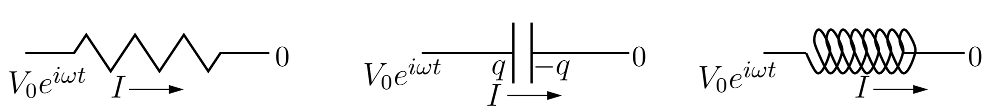
$$R\cdot 0 = V_0 - e^{-(L/R)(0+C)} \qquad \text{implies} \qquad e^{-(L/R)(C)} = V_0$$
$$RI(t) = V_0 - V_0 e^{-Lt/R} \qquad \text{or} \qquad I(t) = (1 - e^{-Lt/R})V_0/R$$

and somehow the units have worked themselves out. Logarithms do this, but you still better check. The current in the circuit starts at zero and climbs gradually to its final value $I = V_0/R$.

4.8 Circuits

The methods of section 4.1 apply to simple linear circuits, and the use of complex algebra as in that section leads to powerful and simple ways to manipulate such circuit equations. You probably remember the result of putting two resistors in series or in parallel, but what about combinations of capacitors or inductors under the same circumstances? And what if you have some of each? With the right tools, all of these questions become the same question, so it's not several different techniques, but one.

If you have an oscillating voltage source (a wall plug), and you apply it to a resistor or to a capacitor or to an inductor, what happens? In the first case, $V = IR$ of course, but what about the others? The voltage equation for a capacitor is $V = q/C$ and for an inductor it is $V = L\,dI/dt$. A voltage that oscillates at frequency ω is $V = V_0 \cos\omega t$, but using this trigonometric function forgoes all the advantages that complex exponentials provide. Instead, assume that your voltage source is $V = V_0 e^{i\omega t}$ with the real part understood. Carry this exponential through the calculation, and take the real part only at the end — often you won't even need to do that.



These are respectively

$$V_0 e^{i\omega t} = IR = I_0 e^{i\omega t} R$$
$$V_0 e^{i\omega t} = q/C \implies i\omega V_0 e^{i\omega t} = \dot{q}/C = I/C = I_0 e^{i\omega t}/C$$
$$V_0 e^{i\omega t} = L\dot{I} = i\omega L I = i\omega L I_0 e^{i\omega t}$$

In each case the exponential factor is in common, and you can cancel it. These equations are then

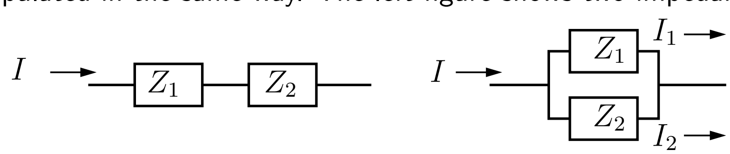
$$V = IR \qquad V = I/i\omega C \qquad V = i\omega L\, I$$

All three of these have the same form: $V =$ (something times)I, and in each case the size of the current is proportional to the applied voltage. The factors of i implies that in the second and third cases the current is $\pm 90^\circ$ out of phase with the voltage cycle.

The coefficients in these equations generalize the concept of resistance, and they are called "impedance," respectively resistive impedance, capacitive impedance, and inductive impedance.

$$V = Z_R I = RI \qquad V = Z_C I = \frac{1}{i\omega C} I \qquad V = Z_L I = i\omega L I \tag{4.39}$$

Impedance appears in the same place as does resistance in the direct current situation, and this implies that it can be manipulated in the same way. The left figure shows two impedances in series.



The total voltage from left to right in the left picture is

$$V = Z_1 I + Z_2 I = (Z_1 + Z_2) I = Z_{\text{total}} I \tag{4.40}$$

It doesn't matter if what's inside the box is a resistor or some more complicated impedance, it matters only that each box obeys $V = ZI$ and that the total voltage from left to right is the sum of the two voltages. Impedances in series add. You don't need the common factor $e^{i\omega t}$.

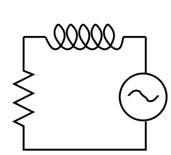
For the second picture, for which the components are in parallel, the voltage is the same on each impedance and charge is conserved, so the current entering the circuit obeys

$$I = I_1 + I_2, \qquad \text{then} \qquad \frac{V}{Z_{\text{total}}} = \frac{V}{Z_1} + \frac{V}{Z_2} \qquad \text{or} \qquad \frac{1}{Z_{\text{total}}} = \frac{1}{Z_1} + \frac{1}{Z_2} \tag{4.41}$$

Impedances in parallel add as reciprocals, so both of these formulas generalize the common equations for resistors in series and parallel. They also include as a special case the formula you may have seen before for adding capacitors in series and parallel.

In the example Eq. (4.37), if you replace the constant voltage by an oscillating voltage, you have two impedances in series.

$$Z_{\text{tot}} = Z_R + Z_L = R + i\omega L \implies I = V/(R + i\omega L)$$



What happened to the $e^{-Lt/R}$ term of the previous solution? This impedance manipulation tells you the *inhomogeneous* solution; you still must solve the homogeneous part of the differential equation and add that.

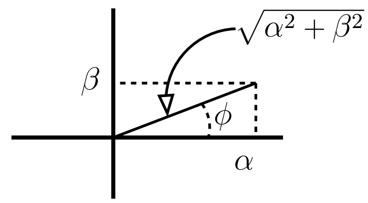
$$L\frac{dI}{dt} + IR = 0 \implies I(t) = Ae^{-Rt/L}$$

The total solution is the sum

$$\begin{aligned} I(t) &= Ae^{-Rt/L} + V_0 e^{i\omega t}\frac{1}{R + i\omega L} \\ \text{real part} &= Ae^{-Rt/L} + V_0\frac{\cos(\omega t - \phi)}{\sqrt{R^2 + \omega^2 L^2}} \qquad \text{where} \qquad \phi = \tan^{-1}\frac{\omega L}{R} \end{aligned} \tag{4.42}$$

How did that last manipulation come about? Change the complex number $R + i\omega L$ in the denominator from rectangular to polar form. Then the division of the complex numbers becomes easy. The dying exponential is called the "transient" term, and the other term is the "steady-state" term.

The denominator is



$$R + i\omega L = \alpha + i\beta = \sqrt{\alpha^2 + \beta^2}\,\frac{\alpha + i\beta}{\sqrt{\alpha^2 + \beta^2}} \tag{4.43}$$

The reason for this multiplication and division by the same factor is that it makes the final fraction have magnitude one. That allows me to write it as an exponential, $e^{i\phi}$. From the picture, the cosine and the sine of the angle ϕ are the two terms in the fraction.

$$\alpha + i\beta = \sqrt{\alpha^2 + \beta^2}\,(\cos\phi + i\sin\phi) = \sqrt{\alpha^2 + \beta^2}\,e^{i\phi} \qquad \text{and} \qquad \tan\phi = \beta/\alpha$$

In summary,

$$V = IZ \;\longrightarrow\; I = \frac{V}{Z} \;\longrightarrow\; Z = |Z|e^{i\phi} \;\longrightarrow\; I = \frac{V}{\sqrt{R^2 + \omega^2 L^2}e^{i\phi}}$$

To satisfy initial conditions, you need the parameter A, but you also see that it gives a dying exponential. After some time this transient term will be negligible, and only the oscillating steady-state term is left. That is what this impedance idea provides.

In even the simplest circuits such as these, that fact that Z is complex implies that the applied voltage is out of phase with the current. $Z = |Z|e^{i\phi}$, so $I = V/Z$ has a phase change of $-\phi$ from V.

What if you have more than one voltage source, perhaps the second having a different frequency from the first? Remember that you're just solving an inhomogeneous differential equation, and you are using the methods of section 4.2. If the external force in Eq. (4.12) has two terms, you can handle them separately then add the results.

4.9 Simultaneous Equations

What's this doing in a chapter on differential equations? Patience. Solve two equations in two unknowns:

$$\begin{aligned} &(\text{X})\;\; ax + by = e \\ &(\text{Y})\;\; cx + dy = f \end{aligned} \qquad d\times(\text{X}) - b\times(\text{Y}): \qquad \begin{aligned} adx + bdy - bcx - bdy &= ed - fb \\ (ad - bc)x &= ed - fb \end{aligned}$$

Similarly, multiply (Y) by a and (X) by c and subtract:

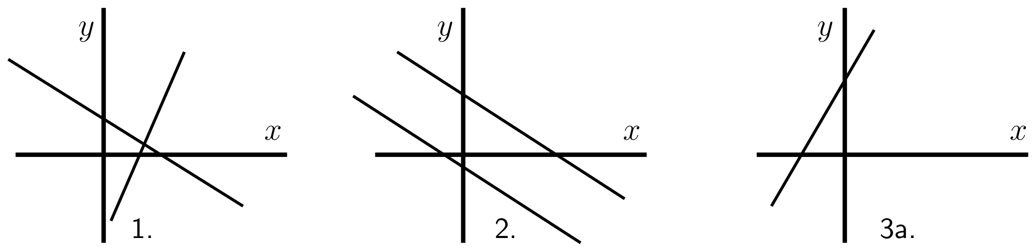
$$\begin{aligned} acx + ady - acx - cby &= fa - ec \\ (ad - bc)y &= fa - ec \end{aligned}$$

Divide by the factor on the left side and you have

$$x = \frac{ed - fb}{ad - bc}, \qquad y = \frac{fa - ec}{ad - bc} \tag{4.44}$$

provided that $ad - bc \neq 0$. This expression appearing in both denominators is the determinant of the equations.

Classify all the essentially different cases that can occur with this simple-looking set of equations and draw graphs to illustrate them. If this looks like problem 1.23, it should.



1. The solution is just as in Eq. (4.44) above and nothing goes wrong. There is exactly one solution. The two graphs of the two equations are two intersecting straight lines.

2. The denominator, the determinant, is zero and the numerator isn't. This is impossible and there are no solutions. When the determinant vanishes, the two straight lines are parallel and the fact that the numerator isn't zero implies that the two lines are distinct and never intersect. (This could also happen if in one of the equations, say (X), $a = b = 0$ and $e \neq 0$. For example $0 = 1$. This obviously makes no sense.)

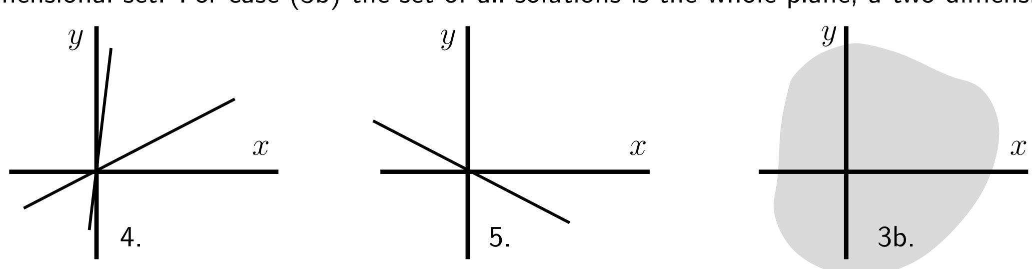
3a. The determinant is zero and so are both numerators. In this case the two lines are not only parallel, they are the same line. The two equations are not really independent and you have an infinite number of solutions.

3b. You can get zero over zero another way. Both equations (X) and (Y) are $0 = 0$. This sounds trivial, but it can really happen. *Every* x and y will satisfy the equation.

4. Not strictly a different case, but sufficiently important to discuss it separately: suppose the the right-hand sides of (X) and (Y) are zero, $e = f = 0$. If the determinant is non-zero, there is a unique solution and it is $x = 0$, $y = 0$.

5. With $e = f = 0$, if the determinant is zero, the two equations are the same equation and there are an infinite number of non-zero solutions.

In the important case for which $e = f = 0$ and the determinant is zero, there are two cases: (3b) and (5). In the latter case there is a one-parameter family of solutions and in the former case there is a two-parameter family. Put another way, for case (5) the set of all solutions is a straight line, a one-dimensional set. For case (3b) the set of all solutions is the whole plane, a two-dimensional set.



Example: consider the two equations

$$kx + (k-1)y = 0, \qquad (1-k)x + (k-1)^2 y = 0$$

For whatever reason, I would like to get a non-zero solution for x and y. Can I? The condition depends on the determinant, so take the determinant and set it equal to zero.

$$k(k-1)^2 - (1-k)(k-1) = 0, \qquad \text{or} \qquad (k+1)(k-1)^2 = 0$$

There are two roots, $k = -1$ and $k = +1$. In the $k = -1$ case the two equations become

$$-x - 2y = 0, \qquad \text{and} \qquad 2x + 4y = 0$$

The second is just -2 times the first, so it isn't a separate equation. The family of solutions is all those x and y satisfying $x = -2y$, a straight line.

In the $k = +1$ case you have

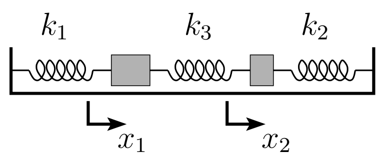
$$x + 0y = 0, \qquad \text{and} \qquad 0 = 0$$

The solution to this is $x = 0$ and $y =$anything and it is again a straight line (the y-axis).

4.10 Simultaneous ODE's

Single point masses are an idealization that has some application to the real world, but there are many more cases for which you need to consider the interactions among many masses. To approach this, take the first step, from one mass to two masses.

Two masses are connected to a set of springs and fastened between two rigid walls as shown. The coordinates for the two masses (moving along a straight line) are x_1 and x_2, and I'll pick the zero point for these coordinates to be the positions at which everything is at equilibrium — no total force on either. When a mass moves away from its equilibrium position there is a force on it. On m_1, the two forces are proportional to the distance by which the two springs k_1 and k_3 are stretched. These two distances are x_1 and $x_1 - x_2$ respectively, so $F_x = ma_x$ applied to each mass gives the equations



$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 - k_3(x_1 - x_2), \qquad \text{and} \qquad m_2\frac{d^2x_2}{dt^2} = -k_2x_2 - k_3(x_2 - x_1) \tag{4.45}$$

I'm neglecting friction simply to keep the algebra down. These are linear, constant coefficient, homogeneous equations, just the same sort as Eq. (4.4) except that there are two of them. What made the solution of (4.4) easy is that the derivative of an exponential is an exponential, so that when you substituted $x(t) = Ae^{\alpha t}$ all that you were left with was an algebraic factor — a quadratic equation in α. Exactly the same method works here.

The only way to find out if this is true is to try it. The big difference is that there are two unknowns instead of one, and the amplitude of the two motions will probably not be the same. If one mass is a lot bigger than the other, you expect it to move less.

Try the solution

$$x_1(t) = Ae^{\alpha t}, \qquad x_2(t) = Be^{\alpha t} \tag{4.46}$$

When you plug this into the differential equations for the masses, all the factors of $e^{\alpha t}$ cancel, just the way it happens in the one variable case.

$$m_1\alpha^2 A = -k_1A - k_3(A - B), \qquad \text{and} \qquad m_2\alpha^2 B = -k_2B - k_3(B - A) \tag{4.47}$$

Rearrange these to put them into a neater form.

$$\begin{aligned} (k_1 + k_3 + m_1\alpha^2)A + \qquad\quad (-k_3)B &= 0 \\ (-k_3)A + (k_2 + k_3 + m_2\alpha^2)B &= 0 \end{aligned} \tag{4.48}$$

The results of problem 1.23 and of section 4.9 tell you all about such equations. In particular, for the pair of equations $ax + by = 0$ and $cx + dy = 0$, the only way to have a non-zero solution for x and y is for the determinant of the coefficients to be zero: $ad - bc = 0$. Apply this result to the problem at hand. Either $A = 0$ and $B = 0$ with a trivial solution *or* the determinant is zero.

$$\big(k_1 + k_3 + m_1\alpha^2\big)\big(k_2 + k_3 + m_2\alpha^2\big) - \big(k_3\big)^2 = 0 \tag{4.49}$$

This is a quadratic equation for α^2, and it determines the frequencies of the oscillation. Note the plural in the word frequencies.

Equation (4.49) is just a quadratic, but it's still messy. For a first example, try a special, symmetric case: $m_1 = m_2 = m$ and $k_1 = k_2$. There's a lot less algebra.

$$\big(k_1 + k_3 + m\alpha^2\big)^2 - \big(k_3\big)^2 = 0 \tag{4.50}$$

You could use the quadratic formula on this, but why? It's already set up to be factored.

$$(k_1 + k_3 + m\alpha^2 - k_3)(k_1 + k_3 + m\alpha^2 + k_3) = 0$$

The product is zero, so one or the other factors is zero. These determine the αs.

$$\alpha_1^2 = -\frac{k_1}{m} \qquad \text{and} \qquad \alpha_2^2 = -\frac{k_1 + 2k_3}{m} \tag{4.51}$$

These are negative, and that's what you should expect. There's no damping and the springs provide restoring forces that should give oscillations. That's just what these imaginary α's provide.

When you examine the equations $ax+by = 0$ and $cx+dy = 0$ the condition that the determinant vanishes is the condition that the two equations are really just one equation, and that the other is not independent of it; it is actually a multiple of the first. You must solve that equation for x and y. Here, arbitrarily pick the first of the equations (4.48) and find the relation between A and B.

$$\alpha_1^2 = -\frac{k_1}{m} \quad\Longrightarrow\quad \big(k_1 + k_3 + m(-(k_1/m))\big)A + \big(-k_3\big)B = 0 \quad\Longrightarrow\quad B = A$$

$$\alpha_2^2 = -\frac{k_1 + 2k_3}{m} \quad\Longrightarrow\quad \big(k_1 + k_3 + m(-(k_1 + 2k_3/m))\big)A + \big(-k_3\big)B = 0 \quad\Longrightarrow\quad B = -A$$

For the first case, $\alpha_1 = \pm i\omega_1 = \pm i\sqrt{k_1/m}$, there are two solutions to the original differential equations. These are called "normal modes."

$$\begin{aligned} x_1(t) &= A_1 e^{i\omega_1 t} \\ x_2(t) &= A_1 e^{i\omega_1 t} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1(t) &= A_2 e^{-i\omega_1 t} \\ x_2(t) &= A_2 e^{-i\omega_1 t} \end{aligned}$$

The other frequency has the corresponding solutions

$$\begin{aligned} x_1(t) &= A_3 e^{i\omega_2 t} \\ x_2(t) &= -A_3 e^{i\omega_2 t} \end{aligned} \qquad \text{and} \qquad \begin{aligned} x_1(t) &= A_4 e^{-i\omega_2 t} \\ x_2(t) &= -A_4 e^{-i\omega_2 t} \end{aligned}$$

The total solution to the differential equations is the sum of all four of these.

$$\begin{aligned} x_1(t) &= A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} + A_3 e^{i\omega_2 t} + A_4 e^{-i\omega_2 t} \\ x_2(t) &= A_1 e^{i\omega_1 t} + A_2 e^{-i\omega_1 t} - A_3 e^{i\omega_2 t} - A_4 e^{-i\omega_2 t} \end{aligned} \tag{4.52}$$

The two second order differential equations have four arbitrary constants in their solution. You can specify the initial values of two positions and of two velocities this way. As a specific example suppose that all initial velocities are zero and that the first mass is pushed to coordinate x_0 and released.

$$\begin{aligned} x_1(0) &= x_0 = A_1 + A_2 + A_3 + A_4 \\ x_2(0) &= 0 = A_1 + A_2 - A_3 - A_4 \\ v_{x1}(0) &= 0 = i\omega_1 A_1 - i\omega_1 A_2 + i\omega_2 A_3 - i\omega_2 A_4 \\ v_{x2}(0) &= 0 = i\omega_1 A_1 - i\omega_1 A_2 - i\omega_2 A_3 + i\omega_2 A_4 \end{aligned} \tag{4.53}$$

With a little thought (i.e. don't plunge blindly ahead) you can solve these easily.

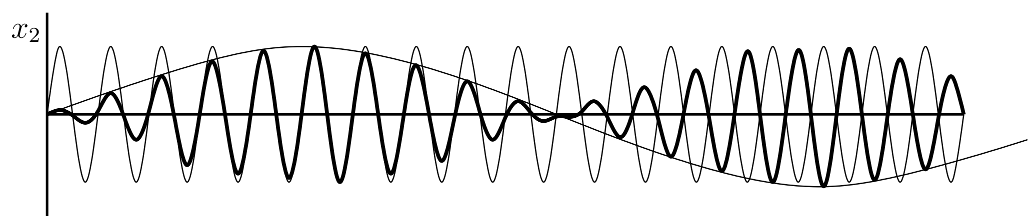
$$A_1 = A_2 = A_3 = A_4 = \frac{x_0}{4}$$

$$\begin{aligned} x_1(t) &= \frac{x_0}{4}\left[e^{i\omega_1 t} + e^{-i\omega_1 t} + e^{i\omega_2 t} + e^{-i\omega_2 t}\right] = \frac{x_0}{2}\left[\cos\omega_1 t + \cos\omega_2 t\right] \\ x_2(t) &= \frac{x_0}{4}\left[e^{i\omega_1 t} + e^{-i\omega_1 t} - e^{i\omega_2 t} - e^{-i\omega_2 t}\right] = \frac{x_0}{2}\left[\cos\omega_1 t - \cos\omega_2 t\right] \end{aligned}$$

From the results of problem 3.34, you can rewrite these as

$$\begin{aligned} x_1(t) &= x_0 \cos\left(\frac{\omega_2 + \omega_1}{2} t\right) \cos\left(\frac{\omega_2 - \omega_1}{2} t\right) \\ x_2(t) &= x_0 \sin\left(\frac{\omega_2 + \omega_1}{2} t\right) \sin\left(\frac{\omega_2 - \omega_1}{2} t\right) \end{aligned} \tag{4.54}$$

As usual you have to draw some graphs to understand what these imply. If the center spring k_3 is a lot weaker than the outer ones, then Eq. (4.51) implies that the two frequencies are close to each other and so $|\omega_1 - \omega_2| \ll \omega_1 + \omega_2$. Examine Eq. (4.54) and you see that one of the two oscillating factors oscillate at a much higher frequency than the other. To sketch the graph of x_2 for example you should draw one factor $\left[\sin\left((\omega_2 + \omega_1)t/2\right)\right]$ and the other factor $\left[\sin\left((\omega_2 - \omega_1)t/2\right)\right]$ and graphically multiply them.



The mass m_2 starts without motion and its oscillations gradually build up. Later they die down and build up again (though with reversed phase). Look at the other mass, governed by the equation for $x_1(t)$ and you see that the low frequency oscillation from the $(\omega_2 - \omega_1)/2$ part is big where the one for x_2 is small and vice versa. The oscillation energy moves back and forth from one mass to the other.

4.11 Legendre's Equation

This equation and its solutions appear when you solve electric and gravitational potential problems in spherical coordinates [problem 9.20]. They appear when you study Gauss's method of numerical integration [Eq. (11.27)] and they appear when you analyze orthogonal functions [problem 6.7]. Because it shows up so often it is worth the time to go through the details in solving it.

$$\left[(1 - x^2)y'\right]' + Cy = 0, \qquad \text{or} \qquad (1 - x^2)y'' - 2xy' + Cy = 0 \tag{4.55}$$

Assume a Frobenius solutions about $x = 0$

$$y = \sum_0^\infty a_k x^{k+s}$$

and substitute into (4.55). Could you use an ordinary Taylor series instead? Yes, the point $x = 0$ is not a singular point at all, but it is just as easy (and more systematic and less prone to error) to use the same method in all cases.

$$(1 - x^2)\sum_0^\infty a_k(k+s)(k+s-1)x^{k+s-2} - 2x\sum_0^\infty a_k(k+s)x^{k+s-1} + C\sum_0^\infty a_k x^{k+s} = 0$$

$$\sum_0^\infty a_k(k+s)(k+s-1)x^{k+s-2} +$$

$$\sum_0^\infty a_k\big[-2(k+s) - (k+s)(k+s-1)\big]x^{k+s} + C\sum_0^\infty a_k x^{k+s} = 0$$

$$\sum_{n=-2}^\infty a_{n+2}(n+s+2)(n+s+1)x^{n+s} -$$

$$\sum_{n=0}^\infty a_n\big[(n+s)^2 + (n+s)\big]x^{n+s} + C\sum_{n=0}^\infty a_n x^{n+s} = 0$$

In the last equation you see the usual substitution $k = n + 2$ for the first sum and $k = n$ for the rest. That makes the exponents match across the equation. In the process, I simplified some of the algebraic expressions.

The indicial equation comes from the $n = -2$ term, which appears only once.

$$a_0 s(s-1) = 0, \qquad \text{so} \qquad s = 0,\ 1$$

Now set the coefficient of x^{n+s} to zero, and solve for a_{n+2} in terms of a_n. Also note that s is a non-negative integer, which says that the solution is non-singular at $x = 0$, consistent with the fact that zero is a regular point of the differential equation.

$$a_{n+2} = a_n \frac{(n+s)(n+s+1) - C}{(n+s+2)(n+s+1)} \tag{4.56}$$

$$a_2 = a_0\frac{s(s+1) - C}{(s+2)(s+1)}, \qquad \text{then} \qquad a_4 = a_2\frac{(s+2)(s+3) - C}{(s+4)(s+3)}, \quad \textit{etc.} \tag{4.57}$$

This looks messier than it is. Notice that the only combination of indices that shows up is $n + s$. The index s is 0 or 1, and n is an even number, so the combination $n + s$ covers the non-negative integers: $0,\ 1,\ 2,\ \ldots$

The two solutions to the Legendre differential equation come from the two cases, $s = 0,\ 1$.

$$\begin{aligned} s = 0: \quad & a_0\bigg[1 + \left(\frac{-C}{2}\right)x^2 + \left(\frac{-C}{2}\right)\left(\frac{2\cdot 3 - C}{4\cdot 3}\right)x^4 + \\ & \qquad \left(\frac{-C}{2}\right)\left(\frac{2\cdot 3 - C}{4\cdot 3}\right)\left(\frac{4\cdot 5 - C}{6\cdot 5}\right)x^6 \cdots\bigg] \\ s = 1: \quad & a_0'\left[x + \left(\frac{1\cdot 2 - C}{3\cdot 2}\right)x^3 + \left(\frac{1\cdot 2 - C}{3\cdot 2}\right)\left(\frac{3\cdot 4 - C}{5\cdot 4}\right)x^5 + \cdots\right] \end{aligned} \tag{4.58}$$

and the general solution is a sum of these.

This procedure gives both solutions to the differential equation, one with even powers and one with odd powers. Both are infinite series and are called Legendre Functions. An important point about both of them is that they blow up as $x \to \pm 1$. This fact shouldn't be too surprising, because the differential equation (4.55) has a singular point there.

$$y'' - \frac{2x}{(1+x)(1-x)}y' + \frac{C}{(1+x)(1-x)}y = 0 \tag{4.59}$$

It's a regular singular point, but it is still singular. A detailed calculation in the next section shows that these solutions behave as $\ln(1-x)$ near $x = 1$.

There is an exception! If the constant C is for example $C = 6$, then with $s = 0$ the equations (4.57) are

$$a_2 = a_0\frac{-6}{2}, \qquad a_4 = a_2\frac{6-6}{12} = 0, \qquad a_6 = a_8 = \ldots = 0$$

The infinite series terminates in a polynomial

$$a_0 + a_2x^2 = a_0[1 - 3x^2]$$

This (after a conventional rearrangement) is a Legendre Polynomial,

$$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2}$$

The numerator in Eq. (4.56) for a_{n+2} is $[(n+s)(n+s+1) - C]$. If this happen to equal zero for some value of $n = N$, then $a_{N+2} = 0$ and so then all the rest of $a_{N+4}\ldots$ are zero too. The series is a polynomial. This will happen only for special values of C, such as the value $C = 6$ above. The values of C that have this special property are

$$C = \ell(\ell+1), \qquad \text{for} \qquad \ell = 0,\ 1,\ 2,\ \ldots \tag{4.60}$$

This may be easier to see in the explicit representation, Eq. (4.58). When a numerator equals zero, all the rest that follow are zero too. When $C = \ell(\ell+1)$ for even ℓ, the first series terminates in a polynomial. Similarly for odd ℓ the second series is a polynomial. These are the Legendre polynomials, denoted $P_\ell(x)$, and the conventional normalization is to require that their value at $x = 1$ is one.

$$\begin{array}{l} P_0(x) = 1 \qquad P_1(x) = x \qquad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \\ P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x \qquad P_4(x) = \frac{35}{8}x^4 - \frac{30}{8}x^2 + \frac{3}{8} \end{array} \tag{4.61}$$

The special case for which the series terminates in a polynomial is by far the most commonly used solution to Legendre's equation. You seldom encounter the general solutions as in Eq. (4.58).

A few properties of the P_ℓ are

$$\begin{array}{ll} (a) & \displaystyle\int_{-1}^{1} dx\, P_n(x)P_m(x) = \frac{2}{2n+1}\delta_{nm} \qquad \text{where} \qquad \delta_{nm} = \begin{cases} 1 & \text{if } n = m \\ 0 & \text{if } n \neq m \end{cases} \\ (b) & (n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x) \\ (c) & \displaystyle P_n(x) = \frac{(-1)^n}{2^n n!}\frac{d^n}{dx^n}(1-x^2)^n \\ (d) & P_n(1) = 1 \qquad P_n(-x) = (-1)^n P_n(x) \\ (e) & \displaystyle\left(1 - 2tx + t^2\right)^{-1/2} = \sum_{n=0}^{\infty} t^n P_n(x) \end{array} \tag{4.62}$$

4.12 Asymptotic Behavior

This is a slightly technical subject, but it will come up occasionally in electromagnetism when you dig into the details of boundary value problems. It will come up in quantum mechanics when you solve some of the standard eigenvalue problems that you face near the beginning of the subject. If you haven't come to these yet then you can skip this part for now.

You solve a differential equation using a Frobenius series and now you need to know something about the solution. In particular, how does the solution behave for large values of the argument? All you have in front of you is an infinite series, and it isn't obvious how it will behave far away from the origin. In the line just after Eq. (4.59) it says that these Legendre functions behave as $\ln(1-x)$. How can you tell this from the series in Eq. (4.58)?

There is a theorem that addresses this. Take two functions described by two series:

$$f(x) = \sum^{\infty} a_k x^k \qquad \text{and} \qquad g(x) = \sum^{\infty} b_k x^k$$

It does not matter where the sums start because you are concerned just with the large values of k. The lower limit could as easily be -14 or $+27$ with no change in the result. The ratio test, Eq. (2.8), will determine the radius of convergence of these series, and

$$\left|\frac{a_{k+1}x^{k+1}}{a_k x^k}\right| < C < 1 \quad \text{for large enough } k$$

is enough to insure convergence. The largest x for which this holds defines the radius of convergence, maybe 1, maybe $\infty\ldots.$ Call it R.

Assume that (after some value of k) all the a_k and b_k are positive, then look at the ratio of the ratios,

$$\frac{a_{k+1}/a_k}{b_{k+1}/b_k}$$

If this approaches one, that will tell you only that the radii of convergence of the two series are the same. If it approaches one *very fast*, and if either one of the functions goes to infinity as x approaches the radius of convergence, then it says that the asymptotic behaviors of the functions defined by the series are the same.

$$\text{If } \frac{a_{k+1}/a_k}{b_{k+1}/b_k} - 1 \longrightarrow 0 \text{ as fast as } \frac{1}{k^2}, \text{ and if either } f(x) \text{ or } g(x) \longrightarrow \infty \text{ as } x \to R$$

$$\text{Then} \quad \frac{f(x)}{g(x)} \longrightarrow \text{a constant as } x \to R$$

There are more general ways to state this, but this handles most cases of interest.

Compare these series near $x = 1$.

$$\frac{1}{1-x} = \sum_0^{\infty} x^k, \qquad \text{or} \qquad \ln(1-x) = \sum_1^{\infty} \frac{x^k}{k}, \qquad \text{or}$$

$$(1-x)^{-1/2} = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}(-x)^k \quad (\alpha = -1/2)$$

Even in the third case, the signs of the terms are the same after a while, so this is relevant to the current discussion. The ratio of ratios for the first and second series is

$$\frac{a_{k+1}/a_k}{b_{k+1}/b_k} = \frac{1}{(k+1)/k} = \frac{1}{1+1/k} = 1 - \frac{1}{k} + \cdots$$

These series behave differently as x approaches the radius of convergence ($x \to 1$). But you knew that. The point is to compare an unknown series to a known one.

Applying this theorem requires some fussy attention to detail. You must make sure that the indices in one series correspond exactly to the indices in the other. Take the Legendre series, Eq. (4.56) and compare it to a logarithmic series. Choose $s = 0$ to be specific; then only even powers of x appear. That means that I want to compare it to a series with even powers, and with radius of convergence $= 1$. First try a binomial series such as for $(1-x^2)^\alpha$, but that doesn't work. See for yourself. The logarithm $\ln(1-x^2)$ turns out to be right. From Eq. (4.56) and from the logarithm series,

$$f(x) = \sum_{n \text{ even}}^{\infty} a_n x^n \qquad \text{with} \qquad a_{n+2} = a_n \frac{(n+s)(n+s+1) - C}{(n+s+2)(n+s+1)}$$

$$g(x) = -\ln(1-x^2) = \sum_{1}^{\infty} \frac{x^{2k}}{k} = \sum b_k x^{2k}$$

To make the indices match, let $n = 2k$ in the second series.

$$g(x) = \sum_{n \text{ even}} \frac{x^n}{n/2} = \sum c_n x^n$$

Now look at the ratios.

$$\frac{a_{n+2}}{a_n} = \frac{n(n+1) - C}{(n+2)(n+1)} = \frac{1 + \frac{1}{n} - \frac{C}{n^2}}{1 + \frac{3}{n} + \frac{2}{n^2}} = 1 - \frac{2}{n} + \cdots$$

$$\frac{c_{n+2}}{c_n} = \frac{2/(n+2)}{2/n} = \frac{n}{n+2} = \frac{1}{1+\frac{2}{n}} = 1 - \frac{2}{n} + \cdots$$

These agree to order $1/n$, so the ratio of the ratios differs from one only in order $1/n^2$, satisfying the requirements of the test. This says the the Legendre functions (the ones where the series does not terminate in a polynomial) are logarithmically infinite near $x = \pm 1$. It's a mild infinity, but it is still an infinity. Is this bad? Not by itself, after all the electric potential of a line charge has a logarithm of r in it. Singular solutions aren't necessarily wrong, it just means that you have to look closely at how you are using them.

Exercises

1 What algebraic equations do you have to solve to find the solutions of these differential equations?

$$\frac{d^3x}{dt^3} + a\frac{dx}{dt} + bx = 0, \qquad\qquad \frac{d^{10}z}{du^{10}} - 3z = 0$$

2 These equations are separable, as in section 4.7. Separate them and integrate, solving for the dependent variable, with one arbitrary constant.

$$\frac{dN}{dt} = -\lambda N, \qquad \frac{dx}{dt} = a^2 + x^2, \qquad \frac{dv_x}{dt} = -a\big(1 - e^{-\beta v_x}\big)$$

3 From Eq. (4.40) and (4.41) what are the formulas for putting capacitors or inductors in series and parallel?

Problems

4.1 If the equilibrium position $x = 0$ for Eq. (4.4) is unstable instead of stable, this reverses the sign in front of k. Solve the problem that led to Eq. (4.10) under these circumstances. That is, the damping constant is b as before, and the initial conditions are $x(0) = 0$ and $v_x(0) = v_0$. What is the small time and what is the large time behavior?
Ans: $\big(2mv_0/\sqrt{b^2+4km}\big)e^{-bt/2m}\sinh\left(\sqrt{b^2/4m+k/m}\,t\right)$

4.2 In the damped harmonic oscillator problem, Eq. (4.4), suppose that the damping term is an *anti*-damping term. It has the sign opposite to the one that I used $(+b\,dx/dt)$. Solve the problem with the initial condition $x(0) = 0$ and $v_x(0) = v_0$ and describe the resulting behavior.
Ans: $\big(2mv_0/\sqrt{4km-b^2}\big)e^{bt/2m}\sin\big(\sqrt{4km-b^2}\,t/2m\big)$

4.3 A point mass m moves in one dimension under the influence of a force F_x that has a potential energy $V(x)$. Recall that the relation between these is

$$F_x = -\frac{dV}{dx}$$

Take the specific potential energy $V(x) = -V_0 a^2/(a^2 + x^2)$, where V_0 is positive. Sketch V. Write the equation $F_x = ma_x$. There is an equilibrium point at $x = 0$, and if the motion is over only small distances you can do a power series expansion of F_x about $x = 0$. What is the differential equation now? Keep just the lowest order non-vanishing term in the expansion for the force and solve that equation subject to the initial conditions that at time $t = 0$, $x(0) = x_0$ and $v_x(0) = 0$.
How does the graph of V change as you vary a from small to large values and how does this same change in a affect the behavior of your solution? Ans: $\omega = \sqrt{2V_0/ma^2}$

4.4 The same as the preceding problem except that the potential energy function is $+V_0 a^2/(a^2 + x^2)$.
Ans: $x(t) = x_0\cosh\big(\sqrt{2V_0/ma^2}\,t\big)$ $(|x| < a/4$ or so, depending on the accuracy you want.)

4.5 For the case of the undamped harmonic oscillator and the force Eq. (4.13), start from the beginning and derive the solution subject to the initial conditions that the initial position is zero and the initial velocity is zero. At the end, compare your result to the result of Eq. (4.15) to see if they agree where they should agree.

4.6 Check the dimensions in the result for the forced oscillator, Eq. (4.15).

4.7 Fill in the missing steps in the derivation of Eq. (4.15).

4.8 For the undamped harmonic oscillator apply an extra oscillating force so that the equation to solve is

$$m\frac{d^2x}{dt^2} = -kx + F_{\text{ext}}(t)$$

where the external force is $F_{\text{ext}}(t) = F_0\cos\omega t$. Assume that $\omega \neq \omega_0 = \sqrt{k/m}$.
Find the general solution to the homogeneous part of this problem.
Find a solution for the inhomogeneous case. You can readily guess what sort of function will give you a $\cos\omega t$ from a combination of x and its second derivative.
Add these and apply the initial conditions that at time $t = 0$ the mass is at rest at the origin. Be sure to check your results for plausibility: 0) dimensions; 1) $\omega = 0$; 2) $\omega \to \infty$; 3) t small (not zero). In

each case explain why the result is as it should be.
Ans: $(F_0/m)[-\cos\omega_0 t + \cos\omega t]/(\omega_0^2 - \omega^2)$

4.9 In the preceding problem I specified that $\omega \neq \omega_0 = \sqrt{k/m}$. Having solved it, you know why this condition is needed. Now take the final result of that problem, including the initial conditions, and take the limit as $\omega \to \omega_0$. [What is the *definition* of a derivative?] You did draw a graph of your result didn't you? Ans: $(F_0/2m\omega_0)t\sin\omega_0 t$

4.10 Show explicitly that you can write the solution Eq. (4.7) in any of several equivalent ways,

$$Ae^{i\omega_0 t} + Be^{-i\omega_0 t} = C\cos\omega_0 t + D\sin\omega_0 t = E\cos(\omega_0 t + \phi)$$

I.e. , given A and B, what are C and D, what are E and ϕ? Are there any restrictions in any of these cases?

4.11 In the damped harmonic oscillator, you can have the special (critical damping) case for which $b^2 = 4km$ and for which $\omega' = 0$. Use a series expansion to take the limit of Eq. (4.10) as $\omega' \to 0$. Also graph this solution. What would happen if you took the same limit in Eqs. (4.8) and (4.9), *before* using the initial conditions?

4.12 (**a**) In the limiting solution for the forced oscillator, Eq. (4.16), what is the nature of the result for small time? Expand the solution through order t^2 and understand what you get. Be careful to be consistent in keeping terms to the same order in t.
(**b**) Part (**a**) involved letting β be very large, then examining the behavior for small t. Now reverse the order: What is the first non-vanishing order in t that you will get if you go back to Eq. (4.13), expand that to first non-vanishing order in time, use that for the external force in Eq. (4.12), and find $x(t)$ for small t. Recall that in this example $x(0) = 0$ and $\dot{x}(0) = 0$, so you can solve for $\ddot{x}(0)$ and then for $\dddot{x}(0)$. The two behaviors are very different.

4.13 The undamped harmonic oscillator equation is $d^2x/dt^2 + \omega^2 x = 0$. Solve this by Frobenius series expansion about $t = 0$.

4.14 Check the algebra in the derivation of the $n = 0$ Bessel equation. Explicitly verify that the general expression for a_{2k} in terms of a_0 is correct, Eq. (4.22).

4.15 Work out the Frobenius series solution to the Bessel equation for the $n = 1/2,\ s = -1/2$ case. Graph both solutions, this one and Eq. (4.23).

4.16 Derive the Frobenius series solution to the Bessel equation for the value of $n = 1$. Show that this method doesn't yield a second solution for this case either.

4.17 Try using a Frobenius series method on $y'' + y/x^3 = 0$ around $x = 0$.

4.18 Solve by Frobenius series $x^2u'' + 4xu' + (x^2 + 2)u = 0$. You should be able to recognize the resulting series (after a little manipulation).

4.19 The harmonic oscillator equation, $d^2y/dx^2 + k^2y = 0$, is easy in terms of the variable x. What is this equation if you change variables to $z = 1/x$, getting an equation in such things as d^2y/dz^2. What sort of singularity does this equation have at $z = 0$? And of course, write down the answer for $y(z)$ to see what this sort of singularity can lead to. Graph it.

4.20 Solve by Frobenius series solution about $x = 0$: $y'' + xy = 0$.
Ans: $1 - (x^3/3!) + (1 \cdot 4\, x^6/6!) - (1 \cdot 4 \cdot 7\, x^9/9!) + \cdots$ is one.

4.21 From the differential equation $d^2u/dx^2 = -u$, finish the derivation for c' as in Eq. (4.29). Derive identities for the functions $c(x + y)$ and $s(x + y)$.

4.22 The chain rule lets you take the derivative of the composition of two functions. The function inverse to s is the function f that satisfies $f\big(s(x)\big) = x$. Differentiate this equation with respect to x and derive that f satisfies $df(x)/dx = 1/\sqrt{1 - x^2}$. What is the derivative of the function inverse to c?

4.23 For the differential equation $u'' = +u$ (note the sign change) use the same boundary conditions for two independent solutions that I used in Eq. (4.28). For this new example evaluate c' and s'. Does $c^2 + s^2$ have the nice property that it did in section 4.5? What about $c^2 - s^2$? What are $c(x + y)$ and $s(x + y)$? What is the derivative of the function inverse to s? to c?

4.24 Apply the Green's function method for the force $F_0\big(1 - e^{-\beta t}\big)$ on the harmonic oscillator without damping. Verify that it agrees with the previously derived result, Eq. (4.15). They should match in a special case.

4.25 An undamped harmonic oscillator with natural frequency ω_0 is at rest for time $t < 0$. Starting at time zero there is an added force $F_0 \sin \omega_0 t$. Use Green's functions to find the motion for time $t > 0$, and analyze the solution for both small and large time, determining if your results make sense. Compare the solution to problems 4.9 and 4.11. Ans: $\big(F_0/2m\omega_0^2\big)\big[\sin(\omega_0 t) - \omega_0 t \cos(\omega_0 t)\big]$

4.26 Derive the Green's function analogous to Eq. (4.32) for the case that the harmonic oscillator is damped.

4.27 Radioactive processes have the property that the rate of decay of nuclei is proportional to the number of nuclei present. That translates into the differential equation $dN/dt = -\lambda N$, where λ is a constant depending on the nucleus. At time $t = 0$ there are N_0 nuclei; how many are present at time t later? The half-life is the time in which one-half of the nuclei decay; what is that in terms of λ? Ans: $\ln 2/\lambda$

4.28 (**a**) In the preceding problem, suppose that the result of the decay is another nucleus (the "daughter") that is itself radioactive with its own decay constant λ_2. Call the first one above λ_1. Write the differential equation for the time-derivative of the number, N_2 of this nucleus. You note that N_2 will change for two reasons, so in time dt the quantity dN_2 has two contributions, one is the decrease because of the radioactivity of the daughter, the other an increase due to the decay of the parent. Set up the differential equation for N_2 and you will be able to use the result of the previous problem as input to this; then solve the resulting differential equation for the number of daughter nuclei as a function of time. Assume that you started with none, $N_2(0) = 0$.
(**b**) Next, the "activity" is the total number of *all* types of decays per time. Compute the activity and graph it. For the plot, assume that λ_1 is substantially smaller than λ_2 and plot the total activity as a function of time. Then examine the reverse case, $\lambda_1 \gg \lambda_2$
Ans: $N_0\lambda_1\big[(2\lambda_2 - \lambda_1)e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}\big]/(\lambda_2 - \lambda_1)$

4.29 The "snowplow problem" was made famous by Ralph Agnew: A snowplow starts at 12:00 Noon in a heavy and steady snowstorm. In the first hour it goes 2 miles; in the second hour it goes 1 mile. When did the snowstorm start? Ans: 11:23

4.30 Verify that the equations (4.52) really do satisfy the original differential equations.

4.31 When you use the "dry friction" model Eq. (4.2) for the harmonic oscillator, you can solve the problem by dividing it into pieces. Start at time $t = 0$ and position $x = x_0$ (positive). The initial velocity of the mass m is zero. As the mass is pulled to the left, set up the differential equation and solve it up to the point at which it comes to a halt. Where is that? You can take that as a new initial condition and solve the problem as the mass is pulled to the right until it stops. Where is that? Then keep repeating the process. Instead or further repetition, examine the case for which the coefficient of kinetic friction is small, and determine to lowest order in the coefficient of friction what is the change in the amplitude of oscillation up to the first stop. From that, predict what the amplitude will be after the mass has swung back to the original side and come to its second stop. In this small μ_k approximation, how many oscillations will it undergo until all motion stops. Let $b = \mu_\text{k} F_N$ Ans: Let $t_n = \pi n/\omega_0$, then for $t_n < t < t_{n+1}$, $x(t) = [x_0 - (2n+1)b/k]\cos\omega_0 t + (-1)^n b/k$. Stops when $t \approx \pi k x_0/2\omega_0 b$ *roughly*.

4.32 A mass m is in an undamped one-dimensional harmonic oscillator and is at rest. A constant external force F_0 is applied for the time interval T and is then turned off. What is the motion of the oscillator as a function of time for all $t > 0$? For what value of T is the amplitude of the motion a maximum after the force is turned off? For what values is the motion a minimum? Of course you need an explanation of why you should have been able to anticipate these two results.

4.33 Starting from the solution Eq. (4.52) assume the initial conditions that both masses start from the equilibrium position and that the first mass is given an initial velocity $v_{x1} = v_0$. Find the subsequent motion of the two masses and analyze it.

4.34 If there is viscous damping on the middle spring of Eqs. (4.45) so that each mass feels an extra force depending on their relative velocity, then these equations will be

$$m_1\frac{d^2x_1}{dt^2} = -k_1x_1 - k_3(x_1 - x_2) - b(\dot{x}_1 - \dot{x}_2), \quad \text{and}$$

$$m_2\frac{d^2x_2}{dt^2} = -k_2x_2 - k_3(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1)$$

Solve these subject to the conditions that all initial velocities are zero and that the first mass is pushed to coordinate x_0 and released. Use the same assumption as before that $m_1 = m_2 = m$ and $k_1 = k_2$.

4.35 For the damped harmonic oscillator apply an extra oscillating force so that the equation to solve is

$$m\frac{d^2x}{dt^2} = -b\frac{dx}{dt} - kx + F_\text{ext}(t)$$

where the external force is $F_\text{ext}(t) = F_0e^{i\omega t}$.
(**a**) Find the general solution to the homogeneous part of this problem.
(**b**) Find a solution for the inhomogeneous case. You can readily guess what sort of function will give you an $e^{i\omega t}$ from a combination of x and its first two derivatives.
This problem is easier to solve than the one using $\cos\omega t$, and at the end, to get the solution for the cosine case, all you have to do is to take the real part of your result.

4.36 You can solve the circuit equation Eq. (4.37) more than one way. Solve it by the methods used earlier in this chapter.

4.37 For a second order differential equation you can pick the position and the velocity any way that you want, and the equation then determines the acceleration. Differentiate the equation and you find that the third derivative is determined too.

$$\frac{d^2x}{dt^2} = -\frac{b}{m}\frac{dx}{dt} - \frac{k}{m}x \qquad \text{implies} \qquad \frac{d^3x}{dt^3} = -\frac{b}{m}\frac{d^2x}{dt^2} - \frac{k}{m}\frac{dx}{dt}$$

Assume the initial position is zero, $x(0) = 0$ and the initial velocity is $v_x(0) = v_0$; determine the second derivative at time zero. Now determine the third derivative at time zero. Now differentiate the above equation again and determine the fourth derivative at time zero.
From this, write down the first five terms of the power series expansion of $x(t)$ about $t = 0$.
Compare this result to the power series expansion of Eq. (4.10) to this order.

4.38 Use the techniques of section 4.6, start from the equation $m\, d^2x/dt^2 = F_x(t)$ with *no* spring force or damping. (**a**) Find the Green's function for this problem, that is, what is the response of the mass to a small kick over a small time interval (the analog of Eq. (4.32))? Develop the analog of Eq. (4.34) for this case. Apply your result to the special case that $F_x(t) = F_0$, a constant for time $t > 0$.
(**b**) You know that the solution of this differential equation involves two integrals of $F_x(t)$ with respect to time, so how can this single integral do the same thing? Differentiate this Green's function integral (for arbitrary F_x) twice with respect to time to verify that it really gives what it's supposed to. This is a special case of some general results, problems 15.19 and 15.20.
Ans: $\frac{1}{m}\int_{-\infty}^{t} dt'\, F_x(t')(t - t')$

4.39 A point mass m moves in one dimension under the influence of a force F_x that has a potential energy $V(x)$. Recall that the relation between these is $F_x = -dV/dx$, and take the specific potential energy $V(x) = -V_0 e^{-x^2/a^2}$, where V_0 is positive. Sketch V. Write the equation $F_x = ma_x$. There is an equilibrium point at $x = 0$, and if the motion is over only small distances you can do a power series expansion of F_x about $x = 0$. What is the differential equation now? Keep just the lowest order non-vanishing term in the expansion for the force and solve that equation subject to the initial conditions that at time $t = 0$, $x(0) = x_0$ and $v_x(0) = 0$. As usual, analyze large and small a.

4.40 Solve by Frobenius series methods

$$\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{1}{x}y = 0$$

Ans: $\sum_{k=0}^{\infty}(-1)^k \frac{x^k}{n!(n+1)!}$ is one.

4.41 Find a series solution about $x = 0$ for $y'' + y\sec x = 0$, at least to a few terms.
Ans: $a_0\big[1 - \frac{1}{2}x^2 + 0x^4 + \frac{1}{720}x^6 + \cdots\big] + a_1\big[x - \frac{1}{6}x^3 - \frac{1}{60}x^5 + \cdots\big]$

4.42 Fill in the missing steps in the equations (4.55) to Eq. (4.58).

4.43 Verify the orthogonality relation Eq. (4.62)(**a**) for Legendre polynomials of order $\ell = 0,\, 1,\, 2,\, 3$.

4.44 Start with the function $\big(1 - 2xt + t^2\big)^{-1/2}$. Use the binomial expansion and collect terms to get a power series in t. The coefficients in this series are functions of x. Carry this out at least to the coefficient of t^3 and show that the coefficients are Legendre polynomials. This is called the generating function for the P_ℓ's. It is $\sum_0^\infty P_\ell(x)t^\ell$

4.45 In the equation of problem 4.17, make the change of independent variable $x = 1/z$. Without actually carrying out the solution of the resulting equation, what can you say about solving it?

4.46 Show that Eq. (4.62)(**c**) has the correct value $P_n(1) = 1$ for all n. Note: $(1-x^2) = (1+x)(1-x)$ and you are examining the point $x = 1$.

4.47 Solve for the complete solution of Eq. (4.55) for the case $C = 0$. For this, don't use series methods, but get the closed form solution. Ans: $A\tanh^{-1}x + B$

4.48 Derive the condition in Eq. (4.60). Which values of s correspond to which values of ℓ?

4.49 Start with the equation $y'' + P(x)y' + Q(x)y = 0$ and assume that you have found one solution: $y = f(x)$. Perhaps you used series methods to find it. (**a**) Make the substitution $y(x) = f(x)g(x)$ and deduce a differential equation for g. Let $G = g'$ and solve the resulting first order equation for G. Finally write g as an integral. This is one method (not necessarily the best) to find the second solution to a differential equation.
(**b**) Apply this result to the $\ell = 0$ solution of Legendre's equation to find another way to solve problem 4.47. Ans: $y = f\int dx\,\frac{1}{f^2}\exp - \int P\,dx$

4.50 Treat the damped harmonic oscillator as a two-point boundary value problem.

$$m\ddot{x} + b\dot{x} + kx = 0, \qquad \text{with} \qquad x(0) = 0 \qquad \text{and} \qquad x(T) = d$$

[For this problem, if you want to set $b = k = T = d = 1$ that's o.k.]
(**a**) Assume that m is very small. To a first approximation neglect it and solve the problem.
(**b**) Since you failed to do part (**a**) — it blew up in your face — solve it exactly instead and examine the solution for very small m. Why couldn't you make the approximation of neglecting m? Draw graphs. Welcome to the world of boundary layer theory and singular perturbations. Ans: $x(t) \approx e^{1-t} - e^{1-t/m}$

4.51 Solve the differential equation $\dot{x} = Ax^2(1+\omega t)$ in closed form and compare the series expansion of the result to Eq. (4.25). Ans: $x = \alpha/\big[1 - A\alpha(t + \omega t^2/2)\big]$

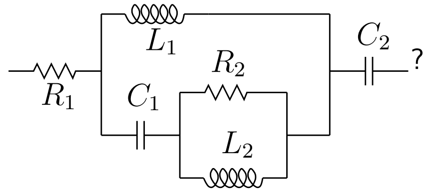
4.52 Solve the same differential equation $\dot{x} = Ax^2(1+\omega t)$ with $x(t_0) = \alpha$ by doing a few iterations of Eq. (4.27).

4.53 Analyze the steady-state part of the solution Eq. (4.42). For the input potential $V_0 e^{i\omega t}$, find the real part of the current explicitly, writing the final form as $I_{\text{max}}\cos(\omega t - \phi)$. Plot I_{max} and ϕ versus ω. Plot $V(t)$ and $I(t)$ on a second graph with time as the axis. Recall these V and I are the real part understood.

4.54 If you have a resistor, a capacitor, and an inductor in series with an oscillating voltage source, what is the steady-state solution for the current? Write the final form as $I_{\text{max}}\cos(\omega t - \phi)$, and plot I_{max} and ϕ versus ω. See what happens if you vary some of the parameters.
Ans: $I = V_0\cos(\omega t - \phi)/|Z|$ where $|Z| = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$ and $\tan\phi = (\omega L - 1/\omega C)/R$

4.55 In the preceding problem, what if the voltage source is a combination of DC and AC, so it is $V(t) = V_0 + V_1\cos\omega t$. Find the steady state solution now.

4.56

What is the total impedance left to right in the circuit 

Ans: $R_1 + (1/i\omega C_2) + 1/\big[(1/i\omega L_1) + 1/\big((1/i\omega C_1) + 1/((1/R_2) + (1/i\omega L_2))\big)\big]$

4.57 Find a Frobenius series solution about $x = 0$ for $y'' + y \csc x = 0$, at least to a few terms.
Ans: $x - \frac{1}{2}x^2 + \frac{1}{12}x^3 - \frac{1}{48}x^4 + \frac{1}{192}x^5 + \cdots$

4.58 Find a series solution about $x = 0$ for $x^2y'' - 2ixy' + (x^2 + i - 1)y = 0$.

4.59 Just as you can derive the properties of the circular and hyperbolic trigonometric functions from the differential equations that they satisfy, you can do the same for the exponential. Take the equation $u' = u$ and consider that solution satisfying the boundary condition $u(0) = 1$.
(**a**) Prove that u satisfies the identity $u(x + y) = u(x)u(y)$.
(**b**) Prove that the function inverse to u has the derivative $1/x$.

4.60 Find the asymptotic behavior of the Legendre series for the $s = 1$ case.

4.61 Find Frobenius series solutions for $xy'' + y = 0$.