Infinite Series

Infinite series are among the most powerful and useful tools that you've encountered in your introductory calculus course. It's easy to get the impression that they are simply a clever exercise in manipulating limits and in studying convergence, but they are among the majors tools used in analyzing differential equations, in developing methods of numerical analysis, in defining new functions, in estimating the behavior of functions, and more.

2.1 The Basics

There are a handful of infinite series that you should memorize and should know just as well as you do the multiplication table. The first of these is the geometric series,

$$1 + x + x^2 + x^3 + x^4 + \cdots = \sum_0^\infty x^n = \frac{1}{1-x} \qquad \text{for } |x| < 1. \tag{2.1}$$

It's very easy derive because in this case you can sum the finite form of the series and then take a limit. Write the series out to the term x^N and multiply it by $(1-x)$.

$$\begin{aligned}(1 + x + x^2 + x^3 + \cdots + x^N)(1-x) = \\ (1 + x + x^2 + x^3 + \cdots + x^N) - (x + x^2 + x^3 + x^4 + \cdots + x^{N+1}) = 1 - x^{N+1}\end{aligned} \tag{2.2}$$

If $|x| < 1$ then as $N \to \infty$ this last term, x^{N+1}, goes to zero and you have the answer. If x is outside this domain the terms of the infinite series don't even go to zero, so there's no chance for the series to converge to anything.

The finite sum up to x^N is useful on its own. For example it's what you use to compute the payments on a loan that's been made at some specified interest rate. You use it to find the pattern of light from a diffraction grating.

$$\sum_0^N x^n = \frac{1 - x^{N+1}}{1-x} \tag{2.3}$$

Some other common series that you need to know are power series for elementary functions:

$$\begin{aligned}
e^x &= 1 + x + \frac{x^2}{2!} + \cdots &&= \sum_0^\infty \frac{x^k}{k!} \\
\sin x &= x - \frac{x^3}{3!} + \cdots &&= \sum_0^\infty (-1)^k \frac{x^{2k+1}}{(2k+1)!} \\
\cos x &= 1 - \frac{x^2}{2!} + \cdots &&= \sum_0^\infty (-1)^k \frac{x^{2k}}{(2k)!} \\
\ln(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots &&= \sum_1^\infty (-1)^{k+1} \frac{x^k}{k} \qquad (|x| < 1) \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)x^2}{2!} + \cdots &&= \sum_{k=0}^\infty \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} x^k \quad (|x| < 1)
\end{aligned} \tag{2.4}$$

Of course, even better than memorizing them is to understand their derivations so well that you can derive them as fast as you can write them down. For example, the cosine is the derivative of the sine, so if you know the latter series all you have to do is to differentiate it term by term to get the cosine series. The logarithm of $(1+x)$ is an integral of $1/(1+x)$ so you can get its series from that of the geometric series. The geometric series is a special case of the binomial series for $\alpha = -1$, but it's easier to remember the simple case separately. You can express all of them as special cases of the general Taylor series.

What is the sine of 0.1 radians? Just use the series for the sine and you have the answer, 0.1, or to more accuracy, $0.1 - 0.001/6 = 0.099833$

What is the square root of 1.1? $\sqrt{1.1} = (1+.1)^{1/2} = 1 + \frac{1}{2}\cdot 0.1 = 1.05$

What is 1/1.9? $1/(2-.1) = 1/[2(1-.05)] = \frac{1}{2}(1+.05) = .5 + .025 = .525$ from the first terms of the geometric series.

What is $\sqrt[3]{1024}$? $\sqrt[3]{1024} = \sqrt[3]{1000+24} = \sqrt[3]{1000(1+24/1000)} =$
$10(1+24/1000)^{1/3} = 10(1+8/1000) = 10.08$

As you see from the last two examples you have to cast the problem into a form fitting the expansion that you know. When you want to use the binomial series, rearrange and factor your expression so that you have

$$\big(1 + \text{something small}\big)^{\alpha}$$

2.2 Deriving Taylor Series

How do you derive these series? The simplest way to get any of them is to assume that such a series exists and then to deduce its coefficients in sequence. Take the sine for example, assume that you can write

$$\sin x = A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots$$

Evaluate this at $x = 0$ to get

$$\sin 0 = 0 = A + B0 + C0^2 + D0^3 + E0^4 + \cdots = A$$

so the first term, $A = 0$. Now differentiate the series, getting

$$\cos x = B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots$$

Again set $x = 0$ and all the terms on the right except the first one vanish.

$$\cos 0 = 1 = B + 2C0 + 3D0^2 + 4E0^3 + \cdots = B$$

Keep repeating this process, evaluating in turn all the coefficients of the assumed series.

$$\begin{aligned}
\sin x &= A + Bx + Cx^2 + Dx^3 + Ex^4 + \cdots & \sin 0 &= 0 = A \\
\cos x &= B + 2Cx + 3Dx^2 + 4Ex^3 + \cdots & \cos 0 &= 1 = B \\
-\sin x &= 2C + 6Dx + 12Ex^2 + \cdots & -\sin 0 &= 0 = 2C \\
-\cos x &= 6D + 24Ex + 60Fx^2 + \cdots & -\cos 0 &= -1 = 6D \\
\sin x &= 24E + 120Fx + \cdots & \sin 0 &= 0 = 24E \\
\cos x &= 120F + \cdots & \cos 0 &= 1 = 120F
\end{aligned}$$

This shows the terms of the series for the sine as in Eq. (2.4).

Does this show that the series converges? If it converges does it show that it converges to the sine? No to both. Each statement requires more work, and I'll leave the second one to advanced calculus books. Even better, when you understand the subject of complex variables, these questions about series become much easier to understand.

The generalization to any function is obvious. You match the coefficients in the assumed expansion, and get

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f''''(0) + \cdots$$

You don't have to do the expansion about the point zero. Do it about another point instead.

$$f(t) = f(t_0) + (t - t_0)f'(t_0) + \frac{(t - t_0)^2}{2!}f''(t_0) + \cdots \tag{2.5}$$

What good are infinite series?
This is sometimes the way that a new function is introduced and developed, typically by determining a series solution to a new differential equation. (Chapter 4)
This is a tool for the numerical evaluation of functions.
This is an essential tool to understand and invent numerical algorithms for integration, differentiation, interpolation, and many other common numerical methods. (Chapter 11)
To understand the behavior of complex-valued functions of a complex variable you will need to understand these series for the case that the variable is a complex number. (Chapter 14)

All the series that I've written above are power series (Taylor series), but there are many other possibilities.

$$\zeta(z) = \sum_1^\infty \frac{1}{n^z} \tag{2.6}$$

$$x^2 = \frac{L^2}{3} + \frac{4L^2}{\pi^2}\sum_1^\infty (-1)^n \frac{1}{n^2}\cos\left(\frac{n\pi x}{L}\right) \qquad (-L \le x \le L) \tag{2.7}$$

The first is a Dirichlet series defining the Riemann zeta function, a function that appears in statistical mechanics among other places.
The second is an example of a Fourier series. See chapter five for more of these.
Still another type of series is the Frobenius series, useful in solving differential equations: its form is $\sum_k a_k x^{k+s}$. The number s need not be either positive or an integer. Chapter four has many examples of this form.

There are a few technical details about infinite series that you have to go through. In introductory calculus courses there can be a tendency to let these few details overwhelm the subject so that you are left with the impression that that's all there is, not realizing that this stuff is useful. Still, you do need to understand it.*

2.3 Convergence

Does an infinite series converge? Does the limit as $N \to \infty$ of the sum, $\sum_1^N u_k$, exist? There are a few common and useful ways to answer this. The first and really the foundation for the others is the comparison test.

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* For animations showing how fast some of these power series converge, check out
www.physics.miami.edu/nearing/mathmethods/power-animations.html

Let u_k and v_k be sequences of real numbers, positive at least after some value of k. Also assume that for all k greater than some finite value, $u_k \le v_k$. Also assume that the sum, $\sum_k v_k$ *does* converge. The other sum, $\sum_k u_k$ then converges too. This is almost obvious, but it's worth the little effort that a proof takes.

The required observation is that an increasing sequence of real numbers, bounded above, has a limit.

After some point, $k = M$, all the u_k and v_k are positive and $u_k \le v_k$. The sum $a_n = \sum_M^n v_k$ then forms an increasing sequence of real numbers, so by assumption this has a limit (the series converges). The sum $b_n = \sum_M^n u_k$ is an increasing sequence of real numbers also. Because $u_k \le v_k$ you immediately have $b_n \le a_n$ for all n.

$$b_n \le a_n \le \lim_{n\to\infty} a_n$$

this simply says that the increasing sequence b_n has an upper bound, so it has a limit and the theorem is proved.

Ratio Test

To apply this comparison test you need a stable of known convergent series. One that you do have is the geometric series, $\sum_k x^k$ for $|x| < 1$. Let this x^k be the v_k of the comparison test. Assume at least after some point $k = K$ that all the $u_k > 0$.
Also that $u_{k+1} \le x u_k$.

$$\text{Then } u_{K+2} \le x u_{K+1} \qquad \text{and} \qquad u_{K+1} \le x u_K \qquad \text{gives} \qquad u_{K+2} \le x^2 u_K$$

You see the immediate extension is

$$u_{K+n} \le x^n u_K$$

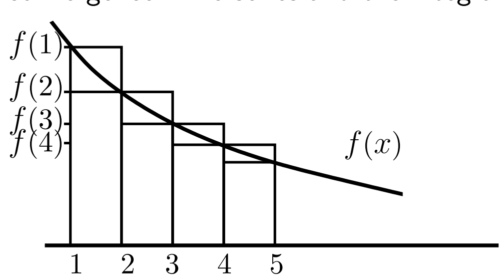
As long as $x < 1$ this is precisely set up for the comparison test using $\sum_n u_K x^n$ as the series that dominates the $\sum_n u_n$. This test, the *ratio test* is more commonly stated for positive u_k as

$$\text{If for large } k, \ \frac{u_{k+1}}{u_k} \le x < 1 \qquad \text{then the series} \qquad \sum u_k \qquad \text{converges} \tag{2.8}$$

This is one of the more commonly used convergence tests, not because it's the best, but because it's simple and it works a lot of the time.

Integral Test

The integral test is another way to check for convergence or divergence. If f is a *decreasing positive* function and you want to determine the convergence of $\sum_n f(n)$, you can look at the integral $\int^\infty dx\, f(x)$ and check *it* for convergence. The series and the integral converge or diverge together.



From the graph you see that the function f lies between the tops of the upper and the lower rectangles. The area under the curve of f between n and $n+1$ lies between the areas of the two rectangles. That's the reason for the assumption that f is decreasing and positive.

$$f(n)\cdot 1 > \int_n^{n+1} dx\, f(x) > f(n+1)\cdot 1$$

Add these inequalities from $n = k$ to $n = \infty$ and you get

$$f(k) + f(k+1) + \cdots > \int_k^{k+1} + \int_{k+1}^{k+2} + \cdots = \int_k^\infty dx\, f(x)$$
$$> f(k+1) + f(k+2) + \cdots > \int_{k+1}^\infty dx\, f(x) > f \cdots \tag{2.9}$$

The only difference between the infinite series on the left and on the right is one term, so either everything converges or everything diverges.

You can do better than this and use these inequalities to get a quick estimate of the sum of a series that would be too tedious to sum by itself. For example

$$\sum_1^\infty \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \sum_4^\infty \frac{1}{n^2}$$

This last sum lies between two integrals.

$$\int_3^\infty dx\, \frac{1}{x^2} > \sum_4^\infty \frac{1}{n^2} > \int_4^\infty dx\, \frac{1}{x^2} \tag{2.10}$$

that is, between 1/3 and 1/4. Now I'll estimate the whole sum by adding the first three terms explicitly and taking the arithmetic average of these two bounds.

$$\sum_1^\infty \frac{1}{n^2} \approx 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2}\left(\frac{1}{3} + \frac{1}{4}\right) = 1.653 \tag{2.11}$$

The exact sum is more nearly 1.644934066848226, but if you use brute-force addition of the original series to achieve accuracy equivalent to this 1.653 estimation you will need to take about 120 terms. This series converges, but not very fast. See also problem 2.24.

Quicker Comparison Test

There is another way to handle the comparison test that works very easily and quickly (if it's applicable). Look at the terms of the series for large n and see what the approximate behavior of the n^{th} term is. That provides a comparison series. This is better shown by an example:

$$\sum_1^\infty \frac{n^3 - 2n + 1/n}{5n^5 + \sin n}$$

For large n, the numerator is essentially n^3 and the denominator is essentially $5n^5$, so for large n this series is approximately like

$$\sum^\infty \frac{1}{5n^2}$$

More precisely, the ratio of the n^{th} term of this approximate series to that of the first series goes to one as $n \to \infty$. This comparison series converges, so the first one does too. If one of the two series diverges, then the other does too.

Apply the ratio test to the series for e^x.

$$e^x = \sum_0^\infty x^k/k! \qquad \text{so} \qquad \frac{u_{k+1}}{u_k} = \frac{x^{k+1}/(k+1)!}{x^k/k!} = \frac{x}{k+1}$$

As $k\to\infty$ this quotient approaches zero no matter the value of x. This means that the series converges for all x.

Absolute Convergence

If a series has terms of varying signs, that should help the convergence. A series is absolutely convergent if it converges when you replace each term by its absolute value. If it's absolutely convergent then it will certainly be convergent when you reinstate the signs. An example of a series that is convergent but not absolutely convergent is

$$\sum_{k=1}^{\infty}(-1)^{k+1}\frac{1}{k}=1-\frac{1}{2}+\frac{1}{3}-\ldots=\ln(1+1)=\ln 2 \tag{2.12}$$

Change all the minus signs to plus and the series is divergent. (Use the integral test.)

Can you rearrange the terms of an infinite series? Sometimes yes and sometimes no. If a series is convergent but not *absolutely* convergent, then each of the two series, the positive terms and the negative terms, is separately divergent. In this case you can rearrange the terms of the series to converge to anything you want! Take the series above that converges to $\ln 2$. I want to rearrange the terms so that it converges to $\sqrt{2}$. Easy. Just start adding the positive terms until you've passed $\sqrt{2}$. Stop and now start adding negative ones until you're below that point. Stop and start adding positive terms again. Keep going and you can get to any number you want.

$$1+\frac{1}{3}+\frac{1}{5}-\frac{1}{2}+\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\frac{1}{13}-\frac{1}{3}\ etc.$$

2.4 Series of Series

When you have a function whose power series you need, there are sometimes easier ways to the result than a straight-forward attack. Not always, but you should look first. If you need the expansion of e^{ax^2+bx} about the origin you can do a lot of derivatives, using the general form of the Taylor expansion. Or you can say

$$e^{ax^2+bx}=1+(ax^2+bx)+\frac{1}{2}(ax^2+bx)^2+\frac{1}{6}(ax^2+bx)^3+\cdots \tag{2.13}$$

and if you need the individual terms, expand the powers of the binomials and collect like powers of x:

$$1+bx+(a+b^2/2)x^2+(ab+b^3/6)x^3+\cdots$$

If you're willing to settle for an expansion about another point, complete the square in the exponent

$$\begin{aligned}e^{ax^2+bx}&=e^{a(x^2+bx/a)}=e^{a(x^2+bx/a+b^2/4a^2)-b^2/4a}=e^{a(x+b/2a)^2-b^2/4a}=e^{a(x+b/2a)^2}e^{-b^2/4a}\\&=e^{-b^2/4a}\big[1+a(x+b/2a)^2+a^2(x+b/2a)^4/2+\cdots\big]\end{aligned}$$

and this is a power series expansion about the point $x_0=-b/2a$.

What is the power series expansion of the secant? You can go back to the general formulation and differentiate a lot or you can use a combination of two known series, the cosine and the geometric series.

$$\begin{aligned}\sec x=\frac{1}{\cos x}&=\frac{1}{1-\frac{1}{2!}x^2+\frac{1}{4!}x^4+\cdots}=\frac{1}{1-\big[\frac{1}{2!}x^2-\frac{1}{4!}x^4+\cdots\big]}\\&=1+[\;]+[\;]^2+[\;]^3+\cdots\\&=1+\big[\tfrac{1}{2!}x^2-\tfrac{1}{4!}x^4+\cdots\big]+\big[\tfrac{1}{2!}x^2-\tfrac{1}{4!}x^4+\ldots\big]^2+\cdots\\&=1+\tfrac{1}{2!}x^2+\big(-\tfrac{1}{4!}+(\tfrac{1}{2!})^2\big)x^4+\cdots\\&=1+\tfrac{1}{2!}x^2+\tfrac{5}{24}x^4+\cdots\end{aligned} \tag{2.14}$$

This is a geometric series, each of whose terms is itself an infinite series. It still beats plugging into the general formula for the Taylor series Eq. (2.5).

What is $1/\sin^3 x$?

$$\begin{aligned}\frac{1}{\sin^3 x} &= \frac{1}{\big(x - x^3/3! + x^5/5! - \cdots\big)^3} = \frac{1}{x^3\big(1 - x^2/3! + x^4/5! - \cdots\big)^3} \\ &= \frac{1}{x^3\big(1+z\big)^3} = \frac{1}{x^3}(1 - 3z + 6z^2 - \cdots) \\ &= \frac{1}{x^3}(1 - 3(-x^2/3! + x^4/5! - \ldots) + 6(-x^2/3! + x^4/5! - \ldots)^2 \\ &= \frac{1}{x^3} + \frac{1}{2x} + \frac{51x}{360} + \cdots\end{aligned}$$

which is a Frobenius series.

2.5 Power series, two variables

The idea of a power series can be extended to more than one variable. One way to develop it is to use exactly the same sort of brute-force approach that I used for the one-variable case. Assume that there is some sort of infinite series and successively evaluate its terms.

$$f(x,y) = A + Bx + Cy + Dx^2 + Exy + Fy^2 + Gx^3 + Hx^2y + Ixy^2 + Jy^3 \cdots$$

Include all the possible linear, quadratic, cubic, and higher order combinations. Just as with the single variable, evaluate it at the origin, the point $(0,0)$.

$$f(0,0) = A + 0 + 0 + \cdots$$

Now differentiate, but this time you have to do it twice, once with respect to x while y is held constant and once with respect to y while x is held constant.

$$\frac{\partial f}{\partial x}(x,y) = B + 2Dx + Ey + \cdots \qquad \text{then} \qquad \frac{\partial f}{\partial x}(0,0) = B$$
$$\frac{\partial f}{\partial y}(x,y) = C + Ex + 2Fy + \cdots \qquad \text{then} \qquad \frac{\partial f}{\partial y}(0,0) = C$$

Three more partial derivatives of these two equations gives the next terms.

$$\begin{aligned}\frac{\partial^2 f}{\partial x^2}(x,y) &= 2D + 6Gx + 2Hy \cdots \\ \frac{\partial^2 f}{\partial x \partial y}(x,y) &= E + 2Hx + 2Iy \cdots \\ \frac{\partial^2 f}{\partial y^2}(x,y) &= 2F + 2Ix + 6Jy \cdots\end{aligned}$$

Evaluate these at the origin and you have the values of D, E, and F. Keep going and you have all the coefficients.

This is awfully cumbersome, but mostly because the crude notation that I've used. You can make it look less messy simply by choosing a more compact notation. If you do it neatly it's no harder to write the series as an expansion about any point, not just the origin.

$$f(x,y) = \sum_{m,n=0}^{\infty} A_{mn}(x-a)^m(y-b)^n \tag{2.15}$$

Differentiate this m times with respect to x and n times with respect to y, then set $x = a$ and $y = b$. Only one term survives and that is

$$\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(a,b) = m!n!A_{mn}$$

I can use subscripts to denote differentiation so that $\frac{\partial f}{\partial x}$ is f_x and $\frac{\partial^3 f}{\partial x^2 \partial y}$ is f_{xxy}. Then the two-variable Taylor expansion is

$$\begin{aligned} f(x,y) = f(0) + &f_x(0)x + f_y(0)y + \\ &\frac{1}{2}\left[f_{xx}(0)x^2 + 2f_{xy}(0)xy + f_{yy}(0)y^2\right] + \\ &\frac{1}{3!}\left[f_{xxx}(0)x^3 + 3f_{xxy}(0)x^2y + 3f_{xyy}(0)xy^2 + f_{yyy}(0)y^3\right] + \cdots \end{aligned} \tag{2.16}$$

Again put more order into the notation and rewrite the general form using A_{mn} as

$$A_{mn} = \frac{1}{(m+n)!}\left(\frac{(m+n)!}{m!n!}\right)\frac{\partial^{m+n} f}{\partial x^m \partial y^n}(a,b) \tag{2.17}$$

That factor in parentheses is variously called the binomial coefficient or a combinatorial factor. Standard notations for it are

$$\frac{m!}{n!(m-n)!} = {}_mC_n = \binom{m}{n} \tag{2.18}$$

The binomial series, Eq. (2.4), for the case of a positive integer exponent is

$$\begin{aligned} (1+x)^m &= \sum_{n=0}^{m}\binom{m}{n}x^n, \qquad \text{or more symmetrically} \\ (a+b)^m &= \sum_{n=0}^{m}\binom{m}{n}a^n b^{m-n} \end{aligned} \tag{2.19}$$

$$\begin{aligned} &(a+b)^2 = a^2 + 2ab + b^2, \qquad (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3, \\ &(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \quad \text{etc.} \end{aligned}$$

Its relation to combinatorial analysis is that if you ask how many different ways can you choose n objects from a collection of m of them, ${}_mC_n$ is the answer.

2.6 Stirling's Approximation

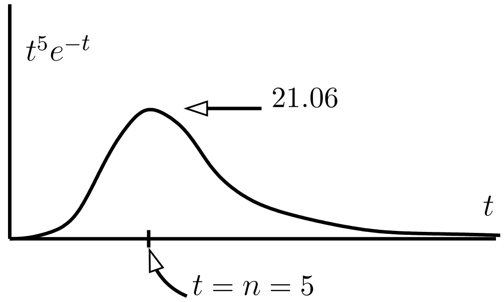
The Gamma function for positive integers is a factorial. A clever use of infinite series and Gaussian integrals provides a useful approximate value for the factorial of large n.

$$n! \sim \sqrt{2\pi n}\, n^n e^{-n} \qquad \text{for large } n \tag{2.20}$$

Start from the Gamma function of $n+1$.

$$n! = \Gamma(n+1) = \int_0^\infty dt\, t^n e^{-t} = \int_0^\infty dt\, e^{-t+n\ln t}$$

The integrand starts at zero, increases, and drops back down to zero as $t \to \infty$. The graph roughly resembles a Gaussian, and I can make this more precise by expanding the exponent around the point where it is a maximum. The largest contribution to the whole integral comes from the region near this point. Differentiate the exponent to find the maximum:



$$\frac{d}{dt}\big(-t + n\ln t\big) = -1 + \frac{n}{t} = 0 \qquad \text{gives} \qquad t = n$$

Expand about this point

$$\begin{aligned} f(t) = -t + n\ln t &= f(n) \qquad + (t-n)f'(n) + (t-n)^2 f''(n)/2 \quad + \cdots \\ &= -n + n\ln n + 0 \qquad\qquad + (t-n)^2(-n/n^2)/2 + \cdots \end{aligned}$$

Keep terms to the second order and the integral is approximately

$$n! \sim \int_0^\infty dt\, e^{-n+n\ln n-(t-n)^2/2n} = n^n e^{-n}\int_0^\infty dt\, e^{-(t-n)^2/2n} \tag{2.21}$$

At the lower limit of the integral, at $t = 0$, this integrand is $e^{-n/2}$, so if n is even moderately large then extending the range of the integral to the whole line $-\infty$ to $+\infty$ won't change the final answer much.

$$n^n e^{-n}\int_{-\infty}^\infty dt\, e^{-(t-n)^2/2n} = n^n e^{-n}\sqrt{2\pi n}$$

where the final integral is just the simplest of the Gaussian integrals in Eq. (1.10).

To see how good this is, try a few numbers

n	n!	Stirling	ratio	difference
1	1	0.922	0.922	0.078
2	2	1.919	0.960	0.081
5	120	118.019	0.983	1.981
10	3628800	3598695.619	0.992	30104.381

You can see that the *ratio* of the exact to the approximate result is approaching one even though the difference is getting very large. This is not a handicap, as there are many circumstances for which this is all you need. This derivation assumed that n is large, but notice that the result is not too bad even for modest values. The error is less than 2% for $n = 5$. There are even some applications, especially in statistical mechanics, in which you can make a still cruder approximation and drop the factor $\sqrt{2\pi n}$. That is because in that context it is the logarithm of $n!$ that appears, and the ratio of the *logarithms* of the exact and even this cruder approximate number goes to one for large n. Try it.

Although I've talked about Stirling's approximation in terms of factorials, it started with the Gamma function, so Eq. (2.20) works just as well for $\Gamma(n+1)$ for any real n:
$\Gamma(11.34 = 10.34 + 1) = 8\,116\,833.918$ and Stirling gives $8\,051\,701$.

Asymptotic
You may have noticed the symbol that I used in Eqs. (2.20) and (2.21). "$\sim$" doesn't mean "approximately equal to" or "about," because as you see here the difference between $n!$ and the Stirling approximation *grows* with n. That the ratio goes to one is the important point here and it gets this special symbol, "asymptotic to."

Probability Distribution
In section 1.4 the equation (1.17) describes the distribution of the results when you toss a coin. It's straight-forward to derive this from Stirling's formula. In fact it is just as easy to do a version of it for which the coin is biased, or more generally, for any case that one of the choices is more likely than the other.

Suppose that the two choices will come up at random with fractions a and b, where $a + b = 1$. You can still picture it as a coin toss, but using a very unfair coin. Perhaps $a = 1/3$ of the time it comes up tails and $b = 2/3$ of the time it comes up heads. If you toss two coins, the possibilities are

TT HT TH HH

and the fractions of the time that you get each pair are respectively

$$a^2 \qquad ba \qquad ab \qquad b^2$$

This says that the fraction of the time that you get no heads, one head, or two heads are

$$a^2 = {}^1\!/_9, \qquad 2ab = {}^4\!/_9, \qquad b^2 = {}^4\!/_9 \qquad \text{with total} \qquad (a+b)^2 = a^2 + 2ab + b^2 = 1 \tag{2.22}$$

Generalize this to the case where you throw N coins at a time and determine how often you expect to see 0, 1, . . . , N heads. Equation (2.19) says

$$(a+b)^N = \sum_{k=0}^{N} \binom{N}{k} a^k b^{N-k} \qquad \text{where} \qquad \binom{N}{k} = \frac{N!}{k!(N-k)!}$$

When you make a trial in which you toss N coins, you expect that the "a" choice will come up N times only the fraction a^N of the trials. All tails and no heads. Compare problem 2.27.

The problem is now to use Stirling's formula to find an approximate result for the terms of this series. This is the fraction of the trials in which you turn up k tails and $N - k$ heads.

$$\begin{aligned} a^k b^{N-k} \frac{N!}{k!(N-k)!} &\sim a^k b^{N-k} \frac{\sqrt{2\pi N}\, N^N e^{-N}}{\sqrt{2\pi k}\, k^k e^{-k} \sqrt{2\pi(N-k)}\, (N-k)^{N-k} e^{-(N-k)}} \\ &= a^k b^{N-k} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{N}{k(N-k)}}\, \frac{N^N}{k^k (N-k)^{N-k}} \end{aligned} \tag{2.23}$$

The complicated parts to manipulate are the factors with all the exponentials of k in them. Pull them out from the denominator for separate handling, leaving the square roots behind.

$$k^k (N-k)^{N-k} a^{-k} b^{-(N-k)}$$

The next trick is to take a logarithm and to do all the manipulations on it.

$$\ln \rightarrow k \ln k + (N-k)\ln(N-k) - k \ln a - (N-k)\ln b = f(k) \tag{2.24}$$

The original function is a maximum when this denominator is a minimum. When the numbers N and k are big, you can treat k as a continuous variable and differentiate with respect to it. Then set this derivative to zero and finally, expand in a power series about that point.

$$\frac{d}{dk}f(k) = \ln k + 1 - \ln(N-k) - 1 - \ln a + \ln b = 0$$

$$\ln\frac{k}{N-k} = \ln\frac{a}{b}, \qquad \frac{k}{N-k} = \frac{a}{b}, \qquad k = aN$$

This should be no surprise; a is the fraction of the time the first choice occurs, and it says that the most likely number of times that it occurs is that fraction times the number of trials. At this point, what is the second derivative?

$$\frac{d^2}{dk^2}f(k) = \frac{1}{k} + \frac{1}{N-k}$$

when $k = aN$, $\qquad f''(k) = \frac{1}{k} + \frac{1}{N-k} = \frac{1}{aN} + \frac{1}{N-aN} = \frac{1}{aN} + \frac{1}{bN} = \frac{1}{abN}$

About this point the power series for $f(k)$ is

$$\begin{aligned} f(k) &= f(aN) + (k-aN)f'(aN) + \frac{1}{2}(k-aN)^2 f''(aN) + \cdots \\ &= N\ln N + \frac{1}{2abN}(k-aN)^2 + \cdots \end{aligned} \tag{2.25}$$

To substitute this back into Eq. (2.23), take its exponential. Then because this will be a fairly sharp maximum, only the values of k near to aN will be significant. That allows me to use this central value of k in the slowly varying square root coefficient of that equation, and I can also neglect higher order terms in the power series expansion there. Let $\delta = k - aN$. The result is the Gaussian distribution.

$$\frac{1}{\sqrt{2\pi}}\sqrt{\frac{N}{aN(N-aN)}} \cdot \frac{N^N}{N^N e^{\delta^2/2abN}} = \frac{1}{\sqrt{2abN\pi}}e^{-\delta^2/2abN} \tag{2.26}$$

When $a = b = 1/2$, this reduces to Eq. (1.17).

When you accumulate N trials at a time (large N) and then look for the distribution in these cumulative results, you will commonly get a Gaussian. This is the central limit theorem, which says that whatever set of probabilities that you start with, not just a coin toss, you will get a Gaussian by averaging the data. (*Not really true.* There are some requirements* on the probabilities that aren't always met, but if as here the variable has a bounded domain then it's o.k. See problems 17.24 and 17.25 for a hint of where a naïve assumption that all distributions behave the same way that Gaussians do can be misleading.) If you listen to the clicks of a counter that records radioactive decays, they sound (and are) random, and the time interval between the clicks varies greatly. If you set the electronics to click at every tenth count, the result will sound regular, and the time interval between clicks will vary only slightly.

* finite mean and variance

2.7 Useful Tricks

There are a variety of ways to manipulate series, and while some of them are simple they are probably not the sort of thing you'd think of until you've seen them once. Example: what is the sum of

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots ?$$

Introduce a parameter that you can manipulate, like the parameter you sometimes introduce to do integrals as in Eq. (1.5). Consider the series with the parameter x in it.

$$f(x)=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+\frac{x^9}{9}-\cdots \tag{2.27}$$

Differentiate this with respect to x to get

$$f'(x)=1-x^2+x^4-x^6+x^8-\cdots$$

That looks a bit like the geometric series except that it has only even powers and the signs alternate. Is that too great an obstacle? As $1/(1-x)$ has only plus signs, then change x to $-x$, and $1/(1+x)$ alternates in sign. Instead of x as a variable, use x^2, then you get exactly what you're looking for.

$$f'(x)=1-x^2+x^4-x^6+x^8-\cdots=\frac{1}{1+x^2}$$

Now to get back to the original series, which is $f(1)$ recall, all that I need to do is integrate this expression for $f'(x)$. The lower limit is zero, because $f(0)=0$.

$$f(1)=\int_0^1 dx\,\frac{1}{1+x^2}=\tan^{-1}x\Big|_0^1=\frac{\pi}{4}$$

This series converges so slowly however that you would never dream of computing π this way. If you take 100 terms, the next term is $1/201$ and you can get a better approximation to π by using $22/7$.

The geometric series is $1+x+x^2+x^3+\cdots$, but what if there's an extra factor in front of each term?

$$f(x)=2+3x+4x^2+5x^3+\cdots$$

Multiply this by x and it is $2x+3x^2+4x^3+5x^4+\cdots$, starting to look like a derivative.

$$xf(x)=2x+3x^2+4x^3+5x^4+\cdots=\frac{d}{dx}\big(x^2+x^3+x^4+\cdots\big)$$

Again, the geometric series pops up, though missing a couple of terms.

$$xf(x)=\frac{d}{dx}\big(1+x+x^2+x^3+\cdots-1-x\big)=\frac{d}{dx}\left[\frac{1}{1-x}-1-x\right]=\frac{1}{(1-x)^2}-1$$

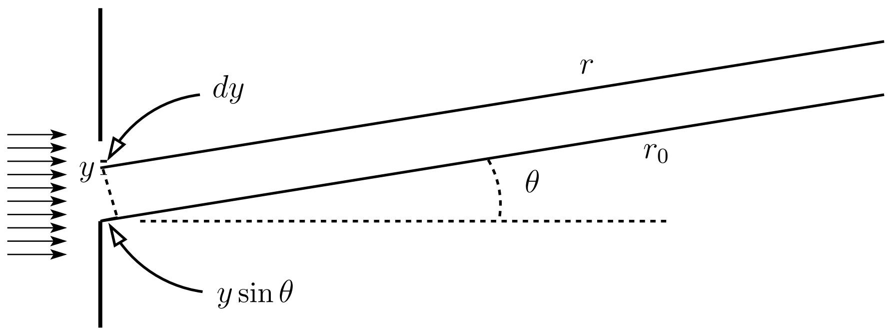
The final result is then

$$f(x)=\frac{1}{x}\left[\frac{1-(1-x)^2}{(1-x)^2}\right]=\frac{2-x}{(1-x)^2}$$

2.8 Diffraction

When light passes through a very small opening it will be diffracted so that it will spread out in a characteristic pattern of higher and lower intensity. The analysis of the result uses many of the tools that you've looked at in the first two chapters, so it's worth showing the derivation first.

The light that is coming from the left side of the figure has a wavelength λ and wave number $k = 2\pi/\lambda$. The light passes through a narrow slit of width $= a$. The Huygens construction for the light that comes through the slit says that you can effectively treat each little part of the slit as if it is a source of part of the wave that comes through to the right. (As a historical note, the mathematical justification for this procedure didn't come until about 150 years after Huygens proposed it, so if you think it isn't obvious why it works, you're right.)



Call the coordinate along the width of the slit y, where $0 < y < a$. I want to find the total light wave that passes through the slit and that heads at the angle θ away from straight ahead. The light that passes through between coordinates y and $y + dy$ is a wave

$$A\,dy\,\cos(kr - \omega t)$$

Its amplitude is proportional to the amplitude of the incoming wave, A, and to the width dy that I am considering. The coordinate along the direction of the wave is r. The total wave that will head in this direction is the sum (integral) over all these little pieces of the slit.

Let r_0 be the distance measured from the bottom of the slit to where the light is received far away. Find the value of r by doing a little trigonometry, getting

$$r = r_0 - y\sin\theta$$

The total wave to be received is now the integral

$$\int_0^a A\,dy\,\cos\big(k(r_0 - y\sin\theta) - \omega t\big) = A\frac{\sin\big(k(r_0 - y\sin\theta) - \omega t\big)}{-k\sin\theta}\bigg|_0^a$$

Put in the limits to get

$$\frac{A}{-k\sin\theta}\big[\sin\big(k(r_0 - a\sin\theta) - \omega t\big) - \sin\big(kr_0 - \omega t\big)\big]$$

I need a trigonometric identity here, one that you can easily derive with the techniques of complex algebra in chapter 3.

$$\sin x - \sin y = 2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right) \tag{2.28}$$

Use this and the light amplitude is

$$\frac{2A}{-k\sin\theta}\sin\left(-\frac{ka}{2}\sin\theta\right)\cos\left(k(r_0-\frac{a}{2}\sin\theta)-\omega t\right) \tag{2.29}$$

The *wave* is the cosine factor. It is a cosine of $(k\cdot\text{distance}-\omega t)$, and the distance in question is the distance to the center of the slit. This is then a wave that appears to be coming from the middle of the slit, but with an amplitude that varies strongly with angle. That variation comes from the other factors in Eq. (2.29).

It's the variation with angle that's important. The intensity of the wave, the power per area, is proportional to the square of the wave's amplitude. I'm going to ignore all the constant factors, so there's no need to worry about the constant of proportionality. The intensity is then (up to a factor)

$$I=\frac{\sin^2\left((ka/2)\sin\theta\right)}{\sin^2\theta} \tag{2.30}$$

For light, the wavelength is about 400 to 700 nm, and the slit may be a millimeter or a tenth of a millimeter. The size of $ka/2$ is then about

$$ka/2=\pi a/\lambda\approx 3\cdot 0.1\,\text{mm}/500\,\text{nm}\approx 1000$$

When you plot this intensity versus angle, the numerator vanishes when the argument of $\sin^2()$ is $n\pi$, with n an integer, $+$, $-$, or 0. This says that the intensity vanishes in these directions *except* for $\theta=0$. In that case the denominator vanishes too, so you have to look closer. For the simpler case that $\theta\neq 0$, these angles are

$$n\pi=\frac{ka}{2}\sin\theta\approx\frac{ka}{2}\theta \qquad n=\pm1,\ \pm2,\ldots$$

Because ka is big, you have many values of n before the approximation that $\sin\theta=\theta$ becomes invalid. You can rewrite this in terms of the wavelength because $k=2\pi/\lambda$.

$$n\pi=\frac{2\pi a}{2\lambda}\theta, \qquad \text{or} \qquad \theta=n\lambda/a$$

What happens at zero? Use power series expansions to evaluate this indeterminate form. The first term in the series expansion of the sine is θ itself, so

$$I=\frac{\sin^2\left((ka/2)\sin\theta\right)}{\sin^2\theta}\longrightarrow\frac{\left((ka/2)\theta\right)^2}{\theta^2}=\left(\frac{ka}{2}\right)^2 \tag{2.31}$$

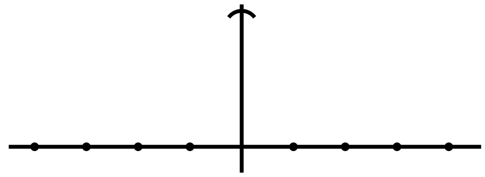
What is the behavior of the intensity *near* $\theta=0$? Again, use power series expansions, but keep another term

$$\sin\theta=\theta-\frac{1}{6}\theta^3+\cdots, \qquad \text{and} \qquad (1+x)^\alpha=1+\alpha x+\cdots$$

Remember, $ka/2$ is big! This means that it makes sense to keep just one term of the sine expansion for $\sin\theta$ itself, but you'd better keep an extra term in the expansion of the $\sin^2(ka\ldots)$.

$$\begin{aligned}I\approx\frac{\sin^2\left((ka/2)\theta\right)}{\theta^2}&=\frac{1}{\theta^2}\left[\left(\frac{ka}{2}\theta\right)-\frac{1}{6}\left(\frac{ka}{2}\theta\right)^3+\cdots\right]^2\\&=\frac{1}{\theta^2}\left(\frac{ka}{2}\theta\right)^2\left[1-\frac{1}{6}\left(\frac{ka}{2}\theta\right)^2+\cdots\right]^2\\&=\left(\frac{ka}{2}\right)^2\left[1-\frac{1}{3}\left(\frac{ka}{2}\theta\right)^2+\cdots\right]\end{aligned}$$

When you use the binomial expansion, put the binomial in the standard form, $(1+x)$ as in the second line of these equations. What is the shape of this function? Forget all the constants, and it looks like $1-\theta^2$. That's a parabola.



The dots are the points where the intensity goes to zero, $n\lambda/a$. Between these directions it reaches a maximum. How big is it there ? These maxima are about halfway between the points where $(ka\sin\theta)/2 = n\pi$. This is

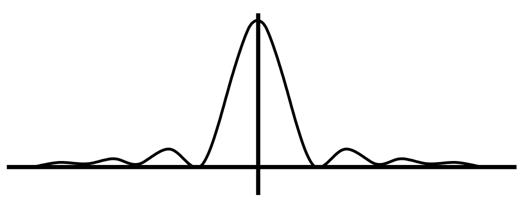
$$\frac{ka}{2}\sin\theta = (n+{}^1\!/\!{}_2)\pi, \qquad n = \pm 1,\, \pm 2,\, \ldots$$

At these angles the value of I is, from Eq. (2.30),

$$I = \left(\frac{ka}{2}\right)^2 \left(\frac{1}{(2n+1)\pi/2}\right)^2$$

The intensity at $\theta = 0$ is by Eq. (2.31), $(ka/2)^2$, so the maxima off to the side have intensities that are smaller than this by factors of

$$\frac{1}{9\pi^2/4} = 0.045, \qquad \frac{1}{25\pi^2/4} = 0.016, \ldots$$



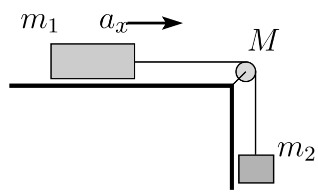
2.9 Checking Results

When you solve any problem, or at least think that you've solved it, you're not done. You still have to check to see whether your result makes any sense. If you are dealing with a problem whose solution is in the back of the book then do you think that the author is infallible? If there is no back of the book and you're working on something that you would like to publish, do you think that *you're* infallible? Either way you can't simply assume that you've made no mistakes; you have to look at your answer skeptically.

There's a second reason, at least as important, to examine your results: that's where you can learn some physics and gain some intuition. Solving a complex problem and getting a complicated answer may involve a lot of mathematics but you don't usually gain any physical insight from doing it. When you analyze your results you can gain an understanding of how the mathematical symbols are related to physical reality. Often an approximate answer to a complicated problem can give you more insight than an exact one, especially if the approximate answer is easier to analyze.

The first tool that you have to use at every opportunity is dimensional analysis. If you are computing a length and your result is a velocity then you are wrong. If you have something in your result that involves adding a time to an acceleration or an angle to a distance, then you've made a mistake; go back and find it. You can do this sort of analysis everywhere, and it is one technique that provides an automatic error finding mechanism. If an equation is dimensionally inconsistent, backtrack a few lines and see whether the units are wrong there too. If they are correct then you know that your error occurred between those two lines; then further narrow the region where the mistake happened by looking for the place at which the dimensions changed from consistent to inconsistent and that's where the mistake happened.

The second tool in your analysis is to examine all the parameters that occur in the result and to see what happens when you vary them. Especially see what happens when you push them to an extreme value. This is best explained by some examples. Start with some simple mechanics to see the procedure.



Two masses are attached by a string of negligible mass and that is wrapped around a pulley of mass M so that it can't slip on the pulley. Analyze them to determine what is wrong with each. Assume that there is no friction between m_1 and the table and that the string does not slip on the pulley.

$$\text{(a) } a_x = \frac{m_2 - m_1}{m_2 + m_1} g \qquad \text{(b) } a_x = \frac{m_2}{m_2 + m_1 - M/2} g \qquad \text{(c) } a_x = \frac{m_2 - M/2}{m_2 + m_1 + M/2} g$$

(a) If $m_1 \gg m_2$, this is negative, meaning that the motion of m_1 is being slowed down. But there's no friction or other such force to do this.
OR If $m_1 = m_2$, this is zero, but there are still unbalanced forces causing these masses to accelerate.

(b) If the combination of masses is just right, for example $m_1 = 1\,\text{kg}$, $m_2 = 1\,\text{kg}$, and $M = 2\,\text{kg}$, the denominator is zero. The expression for a_x blows up — a very serious problem.
OR If M is very large compared to the other masses, the denominator is negative, meaning that a_x is negative and the acceleration is a braking. Without friction, this is impossible.

(c) If $M \gg m_1$ and m_2, the numerator is mostly $-M/2$ and the denominator is mostly $+M/2$. This makes the whole expression negative, meaning that m_1 and m_2 are slowing down. There is no friction to do this, and all the forces are the direction to cause acceleration toward positive x.
OR If $m_2 = M/2$, this equals zero, saying that there is no acceleration, but in this system, a_x will always be positive.

The same picture, but *with* friction μ_k between m_1 and the table.

$$\text{(a) } a_x = \frac{m_2}{m_2 + \mu_\text{k} m_1 + M/2} g \qquad \text{(b) } a_x = \frac{m_2 - \mu_\text{k} m_1}{m_2 - M/2} g \qquad \text{(c) } a_x = \frac{m_2}{m_2 + \mu_\text{k} m_1 - M/2} g$$

(a) If μ_k is very large, this approaches zero. Large friction should cause m_1 to brake to a halt quickly with very large negative a_x.
OR If there is no friction, $\mu_\text{k} = 0$, then m_1 plays no role in this result but if it is big then you know that it will decrease the downward acceleration of m_2.

(b) The denominator can vanish. If $m_2 = M/2$ this is nonsense.

(c) This suffers from both of the difficulties of (a) and (b).

Trajectory Example

When you toss an object straight up with an initial speed v_0, you may expect an answer for the motion as a function of time to be something like

$$v_y(t) = v_0 - gt, \qquad y(t) = v_0 t - \frac{1}{2}gt^2 \tag{2.32}$$

Should you expect this? Not if you remember that there's air resistance. If I claim that the answers are

$$v_y(t) = -v_\mathrm{t} + \big(v_0 + v_\mathrm{t}\big)e^{-gt/v_\mathrm{t}}, \qquad y(t) = -v_\mathrm{t} t + (v_0 + v_\mathrm{t})\frac{v_\mathrm{t}}{g}\big[1 - e^{-gt/v_\mathrm{t}}\big] \tag{2.33}$$

then this claim has to be inspected to see if it makes sense. And I never bothered to tell you what the expression "v_t" means anyway. You have to figure that out. Fortunately that's not difficult in this case. What happens to these equations for very large time? The exponentials go to zero, so

$$v_y \longrightarrow -v_\mathrm{t} + \big(v_0 + v_\mathrm{t}\big) \cdot 0 = -v_\mathrm{t}, \qquad \text{and} \qquad y \longrightarrow -v_\mathrm{t} t + (v_0 + v_\mathrm{t})\frac{v_\mathrm{t}}{g}$$

v_t is the terminal speed. After a long enough time a falling object will reach a speed for which the force by gravity and the force by the air will balance each other and the velocity then remains constant.

Do they satisfy the initial conditions? Yes:

$$v_y(0) = -v_\mathrm{t} + \big(v_0 + v_\mathrm{t}\big)e^0 = v_0, \qquad y(0) = 0 + (v_0 + v_\mathrm{t})\frac{v_\mathrm{t}}{g} \cdot (1 - 1) = 0$$

What do these behave like for small time? They ought to reduce to something like the expressions in Eq. (2.32), but just as important is to determine what the deviation from that simple form is. Keep some extra terms in the series expansion. How many extra terms? If you're not certain, then keep one more than you think you will need. After some experience you will usually be able to anticipate what to do. Expand the exponential:

$$\begin{aligned} v_y(t) &= -v_\mathrm{t} + \big(v_0 + v_\mathrm{t}\big)\left[1 + \frac{-gt}{v_\mathrm{t}} + \frac{1}{2}\left(\frac{-gt}{v_\mathrm{t}}\right)^2 + \cdots\right] \\ &= v_0 - \left(1 + \frac{v_0}{v_\mathrm{t}}\right)gt + \frac{1}{2}\left(1 + \frac{v_0}{v_\mathrm{t}}\right)\frac{g^2t^2}{v_\mathrm{t}} + \cdots \end{aligned}$$

The coefficient of t says that the object is slowing down more rapidly than it would have without air resistance. So far, so good. Is the factor right? Not yet clear, so keep going. Did I need to keep terms to order t^2? Probably not, but there wasn't much algebra involved in doing it, so it was harmless.

Look at the other equation, for y.

$$\begin{aligned} y(t) &= -v_\mathrm{t} t + (v_0 + v_\mathrm{t})\frac{v_\mathrm{t}}{g}\left[1 - \left[1 - \frac{gt}{v_\mathrm{t}} + \frac{1}{2}\left(\frac{gt}{v_\mathrm{t}}\right)^2 - \frac{1}{6}\left(\frac{gt}{v_\mathrm{t}}\right)^3 + \cdots\right]\right] \\ &= v_0 t - \frac{1}{2}\left(1 + \frac{v_0}{v_\mathrm{t}}\right)gt^2 - \frac{1}{6}\left(1 + \frac{v_0}{v_\mathrm{t}}\right)\frac{g^2t^3}{v_\mathrm{t}} + \cdots \end{aligned}$$

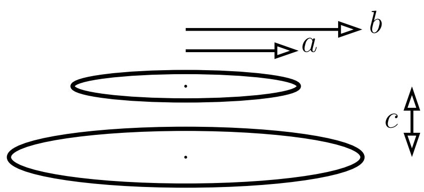
Now differentiate this approximate expression for y with respect to time and you get the approximate expression for v_y. That means that everything appears internally consistent, and I haven't *introduced* any obvious error in the process of approximation.

What if the terminal speed is infinite, so there's no air resistance. The work to answer this is already done. Expanding e^{-gt/v_t} for small time is the same as for large v_t, so you need only look back at the preceding two sets of equations and let $v_\text{t} \longrightarrow \infty$. The result is precisely the equations (2.32), just as you should expect.

You can even determine something about the *force* that I assumed for the air resistance: $F_y = ma_y = m\,dv_y/dt$. Differentiate the approximate expression that you already have for v_y, then at least for small t

$$F_y = m\frac{d}{dt}\left[v_0 - \left(1+\frac{v_0}{v_\text{t}}\right)gt + \frac{1}{2}\left(1+\frac{v_0}{v_\text{t}}\right)\frac{g^2t^2}{v_\text{t}} + \cdots\right]$$
$$= -m\left(1+\frac{v_0}{v_\text{t}}\right)g + \cdots = -mg - mgv_0/v_\text{t} + \cdots \tag{2.34}$$

This says that the force appears to be (1) gravity plus (2) a force proportional to the initial velocity. The last fact comes from the factor v_0 in the second term of the force equation, and at time zero, that *is* the velocity. Does this imply that I assumed a force acting as $F_y = -mg -$ (a constant times)v_y? To this approximation that's the best guess. (It happens to be correct.) To verify it though, you would have to go back to the original un-approximated equations (2.33) and compute the force from them.



Electrostatics Example

Still another example, but from electrostatics this time: Two thin circular rings have radii a and b and carry charges Q_1 and Q_2 distributed uniformly around them. The rings are positioned in two parallel planes a distance c apart and with axes coinciding. The problem is to compute the force of one ring on the other, and for the single non-zero component the answer is (perhaps)

$$F_z = \frac{Q_1Q_2c}{2\pi^2\epsilon_0}\int_0^{\pi/2}\frac{d\theta}{\big[c^2+(b-a)^2+4ab\sin^2\theta\big]^{3/2}}. \tag{2.35}$$

Is this plausible? *First check the dimensions!* The integrand is (dimensionally) $1/(c^2)^{3/2} = 1/c^3$, where c is one of the lengths. Combine this with the factors in front of the integral and one of the lengths (c's) cancels, leaving Q_1Q_2/ϵ_0c^2. This is (again dimensionally) the same as Coulomb's law, $q_1q_2/4\pi\epsilon_0r^2$, so it passes this test.

When you've done the dimensional check, start to consider the parameters that control the result. The numbers a, b, and c can be anything: small, large, or equal in any combination. For some cases you should be able to say what the answer will be, either approximately or exactly, and then check whether this complicated expression agrees with your expectation.

If the rings shrink to zero radius this has $a = b = 0$, so F_z reduces to

$$F_z \to \frac{Q_1Q_2c}{2\pi^2\epsilon_0}\int_0^{\pi/2} d\theta\frac{1}{c^3} = \frac{Q_1Q_2c}{2\pi^2\epsilon_0}\frac{\pi}{2c^3} = \frac{Q_1Q_2}{4\pi\epsilon_0c^2}$$

and this is the correct expression for two point charges a distance c apart.

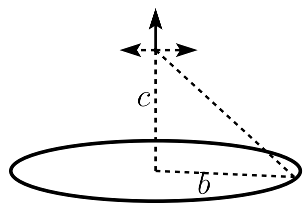
If $c \gg a$ and b then this is really not very different from the preceding case, where a and b are zero.

If $a = 0$ this is

$$F_z \to \frac{Q_1Q_2c}{2\pi^2\epsilon_0}\int_0^{\pi/2}\frac{d\theta}{\big[c^2+b^2\big]^{3/2}} = \frac{Q_1Q_2c}{2\pi^2\epsilon_0}\frac{\pi/2}{\big[c^2+b^2\big]^{3/2}} = \frac{Q_1Q_2c}{4\pi\epsilon_0\big[c^2+b^2\big]^{3/2}} \tag{2.36}$$

The electric field on the axis of a ring is something that you can compute easily. The only component of the electric field at a point on the axis is itself along the axis. You can prove this by assuming that it's false. Suppose that there's a lateral component of $\vec{E}$ and say that it's to the right. Rotate everything by 180° about the axis and this component of $\vec{E}$ will now be pointing in the opposite direction. The ring of charge has not changed however, so $\vec{E}$ must be pointing in the original direction. This supposed sideways component is equal to minus itself, and something that's equal to minus itself is zero.

All the contributions to $\vec{E}$ except those parallel the axis add to zero. Along the axis each piece of charge dq contributes the component



$$\frac{dq}{4\pi\epsilon_0[c^2+b^2]}\cdot\frac{c}{\sqrt{c^2+b^2}}$$

The first factor is the magnitude of the field of the point charge at a distance $r=\sqrt{c^2+b^2}$ and the last factor is the cosine of the angle between the axis and r. Add all the dq together and you get Q_1. Multiply that by Q_2 and you have the force on Q_2 and it agrees with the expressions Eq. (2.36)

If $c\to 0$ then $F_z\to 0$ in Eq. (2.35). The rings are concentric and the outer ring doesn't push the inner ring either up or down.

But wait. In this case, where $c\to 0$, what if $a=b$? Then the force should approach infinity instead of zero because the two rings are being pushed into each other. If $a=b$ then

$$F_z=\frac{Q_1Q_2c}{2\pi^2\epsilon_0}\int_0^{\pi/2}\frac{d\theta}{\left[c^2+4a^2\sin^2\theta\right]^{3/2}} \tag{2.37}$$

If you simply set $c=0$ in this equation you get

$$F_z=\frac{Q_1Q_2 0}{2\pi^2\epsilon_0}\int_0^{\pi/2}\frac{d\theta}{\left[4a^2\sin^2\theta\right]^{3/2}}$$

The numerator is zero, but look at the integral. The variable θ goes from 0 to $\pi/2$, and at the end near zero the integrand looks like

$$\frac{1}{\left[4a^2\sin^2\theta\right]^{3/2}}\approx\frac{1}{\left[4a^2\theta^2\right]^{3/2}}=\frac{1}{8a^3\theta^3}$$

Here I used the first term in the power series expansion of the sine. The integral near the zero end is then approximately

$$\int_0^{\cdots}\frac{d\theta}{\theta^3}=\frac{-1}{2\theta^2}\bigg|_0^{\cdots}$$

and that's infinite. This way to evaluate F_z is indeterminate: $0\cdot\infty$ can be anything. It doesn't show that this F_z gives the right answer, but it doesn't show that it's wrong either.

Estimating a tough integral

Although this is more difficult, even tricky, I'm going to show you how to examine this case for *small* values of c and not for $c=0$. The problem is in figuring out how to estimate the integral (2.37) for small c, and the key is to realize that the only place the integrand gets big is in the neighborhood of $\theta = 0$. The trick then is to divide the range of integration into two pieces

$$\int_0^{\pi/2} \frac{d\theta}{\left[c^2 + 4a^2 \sin^2\theta\right]^{3/2}} = \int_0^{\Lambda} + \int_{\Lambda}^{\pi/2}$$

For any positive value of Λ the second piece of the integral will remain finite even as $c \to 0$. This means that in trying to estimate the way that the whole integral approaches infinity I can ignore the second part of the integral. Now choose Λ small enough that for $0 < \theta < \Lambda$ I can use the approximation $\sin\theta = \theta$, the first term in the series for sine. (Perhaps $\Lambda = 0.1$ or so.)

$$\text{for small } c, \qquad \int_0^{\pi/2} \frac{d\theta}{\left[c^2 + 4a^2 \sin^2\theta\right]^{3/2}} \approx \int_0^{\Lambda} \frac{d\theta}{\left[c^2 + 4a^2\theta^2\right]^{3/2}} + \text{lower order terms}$$

This is an elementary integral. Let $\theta = (c/2a)\tan\phi$.

$$\int_0^{\Lambda} \frac{d\theta}{\left[c^2 + 4a^2\theta^2\right]^{3/2}} = \int_0^{\Lambda'} \frac{(c/2a)\sec^2\phi\, d\phi}{[c^2 + c^2\tan^2\phi]^{3/2}} = \frac{1}{2ac^2}\int_0^{\Lambda'} \cos\phi = \frac{1}{2ac^2}\sin\Lambda'$$

The limit Λ' comes from $\Lambda = (c/2a)\tan\Lambda'$, so this implies $\tan\Lambda' = 2a\Lambda/c$. Now given the tangent of an angle, I want the sine — that's the first page of chapter one.

$$\sin\Lambda' = \frac{2a\Lambda/c}{\sqrt{1 + (2a\Lambda/c)^2}} = \frac{2a\Lambda}{\sqrt{c^2 + 4a^2\Lambda^2}}$$

As $c \to 0$, this approaches one. Put all of this together and you have the behavior of the integral in Eq. (2.37) for small c.

$$\int_0^{\pi/2} \frac{d\theta}{\left[c^2 + 4a^2 \sin^2\theta\right]^{3/2}} \sim \frac{1}{2ac^2} + \text{lower order terms}$$

Insert this into Eq. (2.37) to get

$$F_z \sim \frac{Q_1 Q_2 c}{2\pi^2\epsilon_0} \cdot \frac{1}{2ac^2} = \frac{Q_1 Q_2}{4\pi^2\epsilon_0 a c}$$

Now why should I believe this any more than I believed the original integral? When you are very close to one of the rings, it will look like a long, straight line charge and the linear charge density on it is then $\lambda = Q_1/2\pi a$. What is the electric field of an infinitely long uniform line charge? $E_r = \lambda/2\pi\epsilon_0 r$. So now at the distance c from this line charge you know the E-field and to get the force on Q_2 you simply multiply this field by Q_2.

$$F_z \text{ should be } \frac{\lambda}{2\pi\epsilon_0 c} Q_2 = \frac{Q_1/2\pi a}{2\pi\epsilon_0 c} Q_2 \tag{2.38}$$

and that's exactly what I found in the preceding equation. After all these checks I think that I may believe the result, and more than that you begin to get an intuitive idea of what the result ought to look like. That's at least as valuable. It's what makes the difference between understanding the physics underlying a subject and simply learning how to manipulate the mathematics.

Exercises

1 Evaluate by hand $\cos 0.1$ to four places.

2 In the same way, evaluate $\tan 0.1$ to four places.

3 Use the first two terms of the binomial expansion to estimate $\sqrt{2} = \sqrt{1+1}$. What is the relative error? [(wrong−right)/right]

4 Same as the preceding exercise, but for $\sqrt{1.2}$.

5 What is the domain of convergence for $x - x^4 + x^9 - x^{4^2} + x^{5^2} - \cdots$

6 Does $\displaystyle\sum_{n=0}^{\infty} \cos(n) - \cos(n+1)$ converge?

7 Does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?

8 Does $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^2}$converge?

9 What is the domain of convergence for $\dfrac{x}{1\cdot 2} - \dfrac{x^2}{2\cdot 2^2} + \dfrac{x^3}{3\cdot 3^3} - \dfrac{x^4}{4\cdot 4^4} + \cdots$?

10 From Eq. (2.1), find a series for $\dfrac{1}{(1-x)^2}$.

11 If x is positive, sum the series $1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots$

12 What is the ratio of the exact value of $20!$ to Stirling's approximation for it?

13 For the example in Eq. (2.22), what are the approximate values that would be predicted from Eq. (2.26)?

14 Do the algebra to evaluate Eq. (2.25).

15 Translate this into a question about infinite series and evaluate the two repeating decimal numbers: $0.444444\ldots$, $0.987987987\ldots$

16 What does the integral test tell you about the convergence of the infinite series $\sum_1^\infty n^{-p}$?

17 What would the power series expansion for the sine look like if you require it to be valid in arbitrary units, not just radians? This requires using the constant "C" as in section 1.1.

Problems

2.1 (**a**) If you borrow \$200,000 to buy a house and will pay it back in monthly installments over 30 years at an annual interest rate of 6%, what is your monthly payment and what is the total money that you have paid (neglecting inflation)? To start, you have N payments p with monthly interest i and after all N payments your unpaid balance must reach zero. The initial loan is L and you pay at the end of each month.

$$((L(1+i)-p)(1+i)-p)(1+i)-p \;\cdots\; N \text{ times } = 0$$

Now carry on and find the general expression for the monthly payment. Also find the total paid.
(**b**) Does your general result for arbitrary N reduce to the correct value if you pay everything back at the end of one month? $[L(1+i)=p]$
(**c**) For general N, what does your result become if the interest rate is zero? Ans: \$1199.10, \$431676

2.2 In the preceding problem, suppose that there is an annual inflation of 2%. Now what is the total amount of money that you've paid *in constant dollars?* That is, one hundred dollars in the year 2010 is worth just $\$100/1.02^{10} = \$82.03$ as expressed in year-2000 dollars. Each payment is paid with dollars of gradually decreasing value. Ans: \$324211

2.3 Derive all the power series that you're supposed to memorize, Eq. (2.4).

2.4 Sketch graphs of the functions

$$e^{-x^2} \qquad xe^{-x^2} \qquad x^2e^{-x^2} \qquad e^{-|x|} \qquad xe^{-|x|} \qquad x^2e^{-|x|} \qquad e^{-1/x} \qquad e^{-1/x^2}$$

2.5 The sample series in Eq. (2.7) has a simple graph (x^2 between $-L$ and $+L$) Sketch graphs of one, two, three terms of this series to see if the graph is headed toward the supposed answer.

2.6 Evaluate this same Fourier series for x^2 at $x=L$; the answer is supposed to be L^2. Rearrange the result from the series and show that you can use it to evaluate $\zeta(2)$, Eq. (2.6). Ans: $\pi^2/6$

2.7 Determine the domain of convergence for all the series in Eq. (2.4).

2.8 Determine the Taylor series for $\cosh x$ and $\sinh x$.

2.9 *Working strictly by hand,* evaluate $\sqrt[7]{0.999}$. Also $\sqrt{50}$. Ans: Here's where a calculator can tell you better than I can.

2.10 Determine the next, x^6, term in the series expansion of the secant. Ans: $61x^6/720$

2.11 The power series for the tangent is not as neat and simple as for the sine and cosine. You can derive it by taking successive derivatives as done in the text or you can use your knowledge of the series for the sine and cosine, and the geometric series.

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - x^3/3! + \cdots}{1 - x^2/2! + \cdots} = \left[x - x^3/3! + \cdots\right]\left[1 + (-x^2/2! + \cdots)\right]^{-1}$$

Use the expansion for the geometric series to place all the x^2, x^4, etc. terms into the numerator, treating every term after the "1" as a single small thing. Then collect the like powers to obtain the series at least through x^5.
Ans: $x + x^3/3 + 2x^5/15 + 17x^7/315 + \cdots$

2.12 What is the series expansion for $\csc x = 1/\sin x$? As in the previous problem, use your knowledge of the sine series and the geometric series to get this result at least through x^5. Note: the first term in *this* series is $1/x$. Ans: $1/x + x/6 + 7x^3/360 + 31x^5/15120 + \cdots$

2.13 The exact relativistic expression for the kinetic energy of an object with non-zero mass is

$$K = mc^2(\gamma - 1) \qquad \text{where} \qquad \gamma = \left(1 - v^2/c^2\right)^{-1/2}$$

and c is the speed of light in vacuum. If the speed v is small compared to the speed of light, find an approximate expression for K to show that it reduces to the Newtonian expression for the kinetic energy, but include the next term in the expansion to determine how large the speed v must be in order that this correction term is 10% of the Newtonian expression for the kinetic energy? Ans: $v \approx 0.36\,c$

2.14 Use series expansions to evaluate

$$\lim_{x\to 0} \frac{1 - \cos x}{1 - \cosh x} \qquad \text{and} \qquad \lim_{x\to 0} \frac{\sin kx}{x}$$

2.15 Evaluate using series; you will need both the sine series and the binomial series.

$$\lim_{x\to 0} \left(\frac{1}{\sin^2 x} - \frac{1}{x^2}\right)$$

Now do it again, setting up the algebra differently and finding an easier (or harder) way. Ans: $1/3$

2.16 For some more practice with series, evaluate

$$\lim_{x\to 0} \left(\frac{2}{x} + \frac{1}{1 - \sqrt{1 + x}}\right)$$

Ans: Check experimentally with a few values of x on a pocket calculator.

2.17 Expand the integrand toll find the power series expansion for

$$\ln(1 + x) = \int_0^x dt\,(1 + t)^{-1}$$

Ans: Eq. (2.4)

2.18 (**a**) The error function $\operatorname{erf}(x)$ is defined by an integral. Expand the integrand, integrate term by term, and develop a power series representation for erf. For what values of x does it converge? Evaluate $\operatorname{erf}(1)$ from this series and compare it to the result of problem 1.34. (**b**) Also, as further validation of the integral in problem 1.13, do the power series expansion of both sides of the equation and verify the expansions of the two sides of the equation agree .

2.19 Verify that the combinatorial factor ${}_mC_n$ is really what results for the coefficients when you specialize the binomial series Eq. (2.4) to the case that the exponent is an integer.

2.20 Determine the double power series representation about $(0,0)$ of $1\big/\big[(1 - x/a)(1 - y/b)\big]$

2.21 Determine the double power series representation about $(0,0)$ of $1\big/(1 - x/a - y/b)$

2.22 Use a pocket calculator that can handle 100! and find the ratio of Stirling's approximation to the exact value. You may not be able to find the difference of two such large numbers. An improvement on the basic Stirling's formula is

$$\sqrt{2\pi n}\, n^n e^{-n}\left(1+\frac{1}{12n}\right)$$

What is the ratio of approximate to exact for $n = 1,\ 2,\ 10$?
Ans: $0.99898,\ 0.99948,\ \ldots$

2.23 Evaluate the sum $\sum_1^\infty 1/n(n+1)$. To do this, write the single term $1/n(n+1)$ as a combination of two fractions with denominator n and $(n+1)$ respectively, then start to write out the stated infinite series to a few terms to see the pattern. When you do this you may be tempted to separate it into two series, of positive and of negative terms. Examine the problem of convergence and explain why this is wrong. Ans: 1

2.24 (**a**) You can sometimes use the result of the previous problem to improve the convergence of a slow-converging series. The sum $\sum_1^\infty 1/n^2$ converges, but not very fast. If you add zero to it you don't change the answer, but if you're clever about how you add it you can change this into a much faster converging series. Add $1 - \sum_1^\infty 1/n(n+1)$ to this series and combine the sums. (**b**) After Eq. (2.11) it says that it takes 120 terms to get the stated accuracy. Verify this. For the same accuracy, how many terms does this improved sum take? Ans: about 8 terms

2.25 The electric potential from one point charge is kq/r. For two point charges, you add the potentials of each: $kq_1/r_1 + kq_2/r_2$. Place a charge $-q$ at the origin; place a charge $+q$ at position $(x, y, z) = (0, 0, a)$. Write the total potential from these at an arbitrary position P with coordinates (x, y, z). Now suppose that a is small compared to the distance of P to the origin $\left(r = \sqrt{x^2+y^2+z^2}\right)$ and expand your result to the first non-vanishing power of a, or really of a/r. This is the potential of an electric dipole. Also express your answer in spherical coordinates. See section 8.8 if you need.
Ans: $kqa\,\cos\theta/r^2$

2.26 Do the previous problem, but with charge $-2q$ at the origin and charges $+q$ at each of the two points $(0, 0, a)$ and $(0, 0, -a)$. Again, you are looking for the potential at a point far away from the charges, and up to the lowest non-vanishing power of a. In effect you're doing a series expansion in a/r and keeping the first surviving term. Also express the result in spherical coordinates. The angular dependence should be proportional to $P_2(\cos\theta) = \frac{3}{2}\cos^2\theta - \frac{1}{2}$, a "Legendre polynomial." The r dependence will have a $1/r^3$ in it. This potential is that of a linear quadrupole.

2.27 The combinatorial factor Eq. (2.18) is supposed to be the number of different ways of choosing n objects out of a set of m objects. Explicitly verify that this gives the correct number of ways for $m = 1,\ 2,\ 3,\ 4.$ and all n from zero to m.

2.28 Pascal's triangle is a visual way to compute the values of ${}_mC_n$. Start with the single digit 1 on the top line. Every new line is computed by adding the two neighboring digits on the line above. (At the end of the line, treat the empty space as a zero.)

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

Write the next couple of lines of the triangle and then prove that this algorithm works, that is that the m^{th} row is the ${}_mC_n$, where the top row has $m = 0$. Mathematical induction is the technique that I recommend.

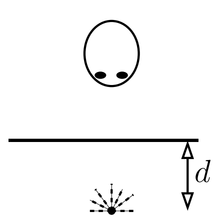
2.29 Sum the series and show
$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots = 1$$

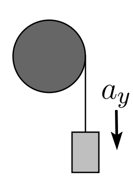
2.30 You know the power series representation for the exponential function, but now apply it in a slightly different context. Write out the power series for the exponential, but with an argument that is a differential operator. The letter h represents some fixed number; interpret the square of d/dx as d^2/dx^2 and find
$$e^{h\frac{d}{dx}} f(x)$$
Interpret the terms of the series and show that the value of this is $f(x+h)$.

2.31 The Doppler effect for sound with a moving source and for a moving observer have different formulas. The Doppler effect for light, including relativistic effects is different still. Show that for low speeds they are all about the same.
$$f' = f\frac{v - v_{\text{o}}}{v} \qquad f' = f\frac{v}{v + v_{\text{s}}} \qquad f' = f\sqrt{\frac{1 - v/c}{1 + v/c}}$$
The symbols have various meanings: v is the speed of sound in the first two, with the other terms being the velocity of the observer and the velocity of the source. In the third equation c is the speed of light and v is the velocity of the observer. And no, $1 = 1$ isn't good enough; you should get these at least to first order in the speed.

2.32 In the equation (2.30) for the light diffracted through a narrow slit, the width of the central maximum is dictated by the angle at the first dark region. How does this angle vary as you vary the width of the slit, a? What is this angle if $a = 0.1\,\text{mm}$ and $\lambda = 700\,\text{nm}$? And how wide will the central peak be on a wall 5 meters from the slit? Take this width to be the distance between the first dark regions on either side of the center.

2.33 An object is a distance d below the surface of a medium with index of refraction n. (For example, water.) When viewed from directly above the object in air (i.e. use small angle approximation), the object appears to be a distance below the surface given by (maybe) one of the following expressions. Show why most of these expressions are implausible; that is, give reasons for eliminating the wrong ones *without* solving the problem explicitly.
$$(1)\ d\sqrt{1+n^2}/n \qquad (2)\ dn/\sqrt{1+n^2} \qquad (3)\ nd \qquad (4)\ d/n \qquad (5)\ dn^2/\sqrt{1+n^2}$$

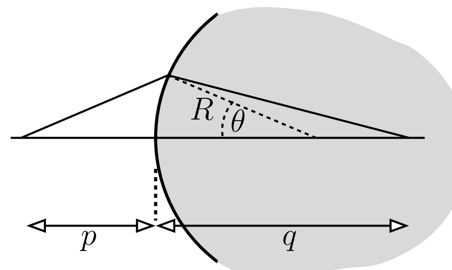




2.34 A mass m_1 hangs from a string that is wrapped around a pulley of mass M. As the mass m_1 falls with acceleration a_y, the pulley rotates. An anonymous source claims that the acceleration of m_1

is one of the following answers. Examine them to determine if any is plausible. That is, examine each and show why it could not be correct. NOTE: solving the problem and then seeing if any of these agree is *not* what this is about.

(1) $a_y = Mg/(m_1 - M)$ (2) $a_y = Mg/(m_1 + M)$ (3) $a_y = m_1 g/M$



2.35 Light travels from a point on the left (p) to a point on the right (q), and on the left it is in vacuum while on the right of the spherical surface it is in glass with an index of refraction n. The radius of the spherical surface is R and you can parametrize the point on the surface by the angle θ from the center of the sphere. Compute the time it takes light to travel on the indicated path (two straight line segments) as a function of the angle θ. Expand the time through second order in a power series in θ and show that the function $T(\theta)$ has a minimum if the distance q is small enough, but that it switches to a maximum when q exceeds a particular value. This position is the focus.

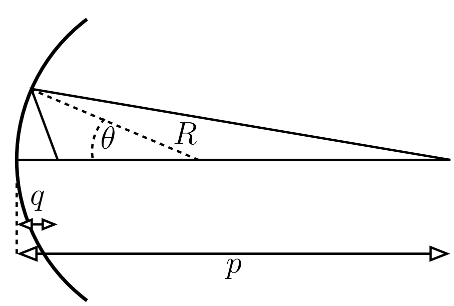
2.36 Combine two other series to get the power series in θ for $\ln(\cos\theta)$.
Ans: $-\frac{1}{2}\theta^2 - \frac{1}{12}\theta^4 - \frac{1}{45}\theta^6 + \cdots$

2.37 Subtract the series for $\ln(1-x)$ and $\ln(1+x)$. For what range of x does this series converge? For what range of arguments of the logarithm does it converge?
Ans: $-1 < x < 1,\ \ 0 < \text{arg} < \infty$

2.38 A function is defined by the integral

$$f(x) = \int_0^x \frac{dt}{1-t^2}$$

Expand the integrand with the binomial expansion and derive the power (Taylor) series representation for f about $x=0$. Also make a hyperbolic substitution to evaluate it in closed form.



2.39 Light travels from a point on the right (p), hits a spherically shaped mirror and goes to a point (q). The radius of the spherical surface is R and you can parametrize the point on the surface by the angle θ from the center of the sphere. Compute the time it takes light to travel on the indicated path (two straight line segments) as a function of the angle θ.
Expand the time through second order in a power series in θ and show that the function $T(\theta)$ has a minimum if the distance q is small enough, but that it switches to a maximum when q exceeds a particular value. This is the focus.

2.40 (**a**) The quadratic equation $ax^2+bx+c=0$ is almost a linear equation if a is small enough: $bx+c=0 \Rightarrow x=-c/b$. You can get a more accurate solution iteratively by rewriting the equation as

$$x = -\frac{c}{b} - \frac{a}{b}x^2$$

Solve this by neglecting the second term, then with this approximate x_1 get an improved value of the root by

$$x_2 = -\frac{c}{b} - \frac{a}{b}x_1^2$$

and you can repeat the process. For comparison take the exact solution and do a power series expansion on it for small a. See if the results agree.
(**b**) Where does the other root come from? That value of x is very large, so the first two terms in the quadratic are the big ones and must nearly cancel. $ax^2 + bx = 0$ so $x = -b/a$. Rearrange the equation so that you can iterate it, and compare the iterated solution to the series expansion of the exact solution.

$$x = -\frac{b}{a} - \frac{c}{ax}$$

Solve $0.001x^2 + x + 1 = 0$. Ans: Solve it exactly and compare.

2.41 Evaluate the limits

$$(\textbf{a})\ \lim_{x\to 0} \frac{\sin x - \tan x}{x}, \qquad (\textbf{b})\ \lim_{x\to 0} \frac{\sin x - \tan x}{x^2}, \qquad (\textbf{c})\ \lim_{x\to 0} \frac{\sin x - \tan x}{x^3}$$

Ans: Check with a pocket calculator for $x = 1.0,\ 0.1,\ 0.01$

2.42 Fill in the missing steps in the derivation of Eq. (2.26).

2.43 Is the result in Eq. (2.26) normalized properly? What is its integral $d\delta$ over all δ? Ans: 1

2.44 A political survey asks 1500 people randomly selected from the entire country whom they will vote for as dog-catcher-in-chief. The results are 49.0% for T.I. Hulk and 51.0% for T.A. Spiderman. Assume that these numbers are representative, an unbiased sample of the electorate. The number $0.49 \times 1500 = aN$ is now your best estimate for the number of votes Mr. Hulk will get in a sample of 1500. Given this estimate, what is the probability that Mr. Hulk will win the final vote anyway?
(**a**) Use Eq. (2.26) to represent this estimate of the probability of his getting various possible outcomes, where the center of the distribution is at $k = aN$. Using $\delta = k - aN$, this probability function is proportional to $\exp\big(-\delta^2/2abN\big)$, and the probability of winning is the sum of all the probabilities of having $k > N/2$, that is, $\int_{N/2}^{\infty} dk$. (**b**) What would the answer be if the survey had asked 150 or 15000 people with the same 49-51 results? Ans: (**a**) $\frac{1}{2}\big[1 - \operatorname{erf}\big(\sqrt{N/2ab}\,(\frac{1}{2} - a)\big)\big]$. 22%, (**b**) 40%, 0.7%

2.45 For the function defined in problem 2.38, what is its behavior near $x = 1$? Compare this result to equation (1.4). Note: the integral is $\int_0^{\Lambda} + \int_{\Lambda}^{x}$. Also, $1 - t^2 = (1+t)(1-t)$, and this $\approx 2(1-t)$ near 1.

2.46 (**a**) What is the expansion of $1/(1+t^2)$ in powers of t for small t. (**b**) That was easy, now what is it for large t? In each case, what is the domain of convergence?

2.47 The "average" of two numbers a and b commonly means $(a+b)/2$, the arithmetic mean. There are many other averages however. $(a, b > 0)$

$$M_n(a,b) = \big[(a^n + b^n)/2\big]^{1/n}$$

is the n^{th} mean, also called the power mean, and it includes many others as special cases. $n = 2$: root-mean-square, $n = -1$: harmonic mean. Show that this includes the geometric mean too: $\sqrt{ab} = \lim_{n\to 0} M_n(a,b)$. It can be shown that $dM_n/dn > 0$; what inequalities does this imply for various means? Ans: harmonic $\le$ geometric $\le$ arithmetic $\le$ rms

2.48 Using the definition in the preceding problem, show that $dM_n/dn > 0$. [Tough!]

2.49 In problem 2.18 you found the power series expansion for the error function — good for small arguments. Now what about large arguments?

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x dt\, e^{-t^2} = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty dt\, e^{-t^2} = 1 - \frac{2}{\sqrt{\pi}}\int_x^\infty dt\, \frac{1}{t}\cdot te^{-t^2}$$

Notice that you *can* integrate the te^{-t^2} factor explicitly, so integrate by parts. Then do it again and again. This provides a series in inverse powers that allows you evaluate the error function for large arguments. What is $\operatorname{erf}(3)$? Ans: 0.9999779095 See Abramowitz and Stegun: 7.1.23.

2.50 A friend of mine got a different result for Eq. (2.35). Instead of $\sin^2\theta$ in the denominator, he got a $\sin\theta$. Analyze his answer for plausibility.

2.51 Find the minimum of the function $f(r) = ar + b/r$ for a, b, $r > 0$. Then find the series expansion of f about that point, at least as far as the first non-constant term.

2.52 In problem 2.15 you found the limit of a function as $x \to 0$. Now find the behavior of the same function as a series expansion for small x, through terms in x^2. Ans: $\frac{1}{3} + \frac{1}{15}x^2$. To test whether this answer or yours or neither is likely to be correct, evaluate the exact and approximate values of this for moderately small x on a pocket calculator.

2.53 Following Eq. (2.34) the tentative conclusion was that the force assumed for the air resistance was a constant times the velocity. Go back the the exact equations (2.33) and compute this force without approximation, showing that it is in fact a constant times the velocity. And of course find the constant.

2.54 An object is thrown straight up with speed v_0. There is air resistance and the resulting equation for the velocity is claimed to be (only while it's going up)

$$v_y(t) = v_\text{t}\frac{v_0 - v_\text{t}\tan(gt/v_\text{t})}{v_\text{t} + v_0\tan(gt/v_\text{t})}$$

where v_t is the terminal speed of the object after it turns around and has then been falling long enough. (**a**) Check whether this equation is plausible by determining if it reduces to the correct result if there is no air resistance and the terminal speed goes to infinity. (**b**) Now, what is the velocity for small time and then use $F_y = ma_y$ to infer the probable speed dependence of what I assumed for the air resistance in deriving this expression. See problem 2.11 for the tangent series. (**c**) Use the exact $v_y(t)$ to show that no matter how large the initial speed is, it will stop in no more than some maximum time. For a bullet that has a terminal speed of 100 m/s, this is about 16 s.

2.55 Under the same circumstances as problem 2.54, the equation for position versus time is

$$y(t) = \frac{v_t^2}{g}\ln\left(\frac{v_\text{t}\cos(gt/v_\text{t}) + v_0\sin(gt/v_\text{t})}{v_\text{t}}\right)$$

(**a**) What is the behavior of this for small time? Analyze and interpret what it says and whether it behaves as it should. (**b**) At the time that it reaches its maximum height ($v_y = 0$), what is its position? Note that you don't need to have an explicit value of t for which this happens; you use the equation that t satisfies.

2.56 You can get the individual terms in the series Eq. (2.13) another way: multiply the two series:

$$e^{ax^2+bx} = e^{ax^2}e^{bx}$$

Do so and compare it to the few terms found after (2.13).