Extrasolar Planets and Astrophysical Discs
Problem Set 3: Solutions

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The donor star is a solar-type star. We can therefore assume that it has the same radius as the Sun, $1\,R_\odot$. The donor star fills its Roche lobe, so therefore the size of its Roche lobe is $1\,R_\odot$. The two stars have the same mass, $1\,M_\odot$, so the centre of mass and the L_1 point lie midway between the two stars. The size of the Roche lobe of the white dwarf therefore must be the same as that of the donor star, $1\,R_\odot$. The accretion disc has a radius 2/3 of the Roche lobe radius. Therefore the disc radius is $R_d = \frac{2}{3} R_\odot = 4.7 \times 10^8$ m.
We must know the number density n of molecules/atoms to calculate the mean free path. (In reality, the particles here will be atoms, because the temperatures are so hot that molecules will dissociate into their constituent atoms.)
The surface density Σ is assumed to be constant across the disc. Therefore Σ will be given by the total mass M_d of the disc divided by its surface area, giving $\Sigma = \dfrac{M_d}{\pi R_d^2}$, where R_d is the radius of the disc. The density in the disc is $\rho \sim \dfrac{\Sigma}{2H}$, where $H(R)$ is the thickness of the disc at a radius R. However, $H(R) = \left(\frac{H(R)}{R}\right) R$. Using the assumption that H/R is constant through the disc (we have $H/R = 0.03$), and assuming that $R \sim R_d$ over much of the disc, we can estimate the density using

$$\rho \sim \frac{\Sigma}{2\left(\frac{H}{R}\right) R_d} \sim \frac{M_d}{2\pi \left(\frac{H}{R}\right) R_d^3} ,$$

on substituting for Σ. Using a mass of the disc at outburst of $M_d = 10^{-11}\,M_\odot = 2 \times 10^{19}$ kg, $R_d = 4.7 \times 10^8$ m and $H/R = 0.03$, we get $\rho \sim 5 \times 10^{-37}\,M_\odot\,\mathrm{m}^{-3} \sim 1 \times 10^{-6}\,\mathrm{kg\,m}^{-3}$. The number density is then $n = \rho/m_H$, where m_H is the mass of the hydrogen atom. This gives, $n \sim 1 \times 10^{-6}/1.67 \times 10^{-27}\,\mathrm{m}^{-3} \sim 6 \times 10^{20}\,\mathrm{m}^{-3}$. So the mean free path is

$$L = \frac{1}{n\sigma} = \frac{1}{n\pi a^2} \sim \frac{1}{6 \times 10^{20}\,\pi\,(10^{-10})^2}\ \mathrm{m} \sim 0.05\ \mathrm{m} ,$$

where a is the radius of the H atom.
The kinematic viscosity is $\nu \sim L c_s$, where c_s is the sound speed in the gas. We have

$$c_s = \sqrt{\frac{\mathcal{R}T}{\mu}} ,$$

where $\mathcal{R}$ is the gas constant, T is the absolute temperature and μ is the mean molecular weight. Use $\mu = 1$ for atomic hydrogen (assuming that the gas is atomic at these temperatures). So

$$c_s = \sqrt{\frac{8.314 \times 6.6 \times 10^4}{0.001}}\ \mathrm{m\,s^{-1}} = 2.3 \times 10^4\ \mathrm{m\,s^{-1}}$$

(note the use of 0.001 kg for the molecular weight in this equation because of the need to convert to S.I. units in which the mass of an Avogadro number of particles is 0.001 kg = 1 g).
The kinematic viscosity is therefore $\nu \sim Lc_s \sim 0.05{\times}2.3{\times}10^4 \mathrm{m^2\,s^{-1}} \sim 1{\times}10^3 \mathrm{m^2\,s^{-1}}$.
The evolutionary timescale will be

$$\tau_{ev} \;\simeq\; \frac{R_d^2}{3\nu} \;\simeq\; \frac{(4.7\times10^8)^2}{3\times1\times10^3}\ \mathrm{s} \;\simeq\; 6\times10^{13}\ \mathrm{s} \;\simeq\; 2\times10^6\ \mathrm{yr}\ .$$

So the timescale implied by molecular viscosity is much too long to account for the $\sim$ 5 day observed variation in brightness.
Another, much greater, source of viscosity is required.
Observations show that $\tau_{ev} \simeq R_D^2/3\nu \simeq$ 5 days $\sim 4.3\times10^3$ s. This gives

$$\nu \;\simeq\; \frac{R_d^2}{3\tau_{ev}} \;\simeq\; \frac{(4.7\times10^8)^2}{3\times4.3\times10^5}\ \mathrm{m^2\,s^{-1}} \;\simeq\; 2\times10^{11}\ \mathrm{m^2\,s^{-1}}\ .$$

The 'alpha' model of turbulence represents the kinematic viscosity as $\nu = \alpha H c_s$. From this we can estimate the α parameter as

$$\alpha \;=\; \frac{\nu}{Hc_s} \;\simeq\; \frac{\nu}{\left(\frac{H}{R}\right) R_d\, c_s} \;\simeq\; \frac{2\times10^{11}}{0.03\times4.7\times10^8\times2.3\times10^4} \;\simeq\; 0.5\ .$$

We require $\alpha \simeq 0.5$ to explain the timescale of the observed outbursts.