H. Solutions to course work 4

Q1. *25 Marks*

Using a locally-inertial coordinate system prove that the Riemann tensor has the following symmetry properties:
a) Permutation within a pair of indices

$$R_{iklm} = -R_{kilm} = -R_{ikml}.$$

Solution:

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}, \tag{H.1}$$

and

$$R_{iklm} = g_{ip}R^p_{klm} = g_{ip}\left(\Gamma^p_{km,l} - \Gamma^p_{kl,m} + \Gamma^p_{nl}\Gamma^n_{km} - \Gamma^p_{nm}\Gamma^n_{kl}\right). \tag{H.2}$$

To prove any tensor identity it is enough to show that this identity is true just in the local Galilean frame of reference. In the local Galilean frame

$$\Gamma^i_{nl} = 0, \ \text{ while } \ \Gamma^p_{km,l} \neq 0,$$

hence

$$R_{iklm} = \eta_{ip}\left(\Gamma^p_{km,l} - \Gamma^p_{kl,m}\right). \tag{H.3}$$

Taking into account that

$$\Gamma^p_{km,l} = \frac{\eta^{pn}}{2}(g_{kn,m} + g_{nm,k} - g_{km,n})_{,l}, \tag{H.4}$$

we have

$$R_{iklm} = \frac{1}{2}\eta_{ip}\left[\eta^{pv}(g_{kv,m} + g_{mv,k} - g_{km,v})_l - \eta^{pv}(g_{kv,l} + g_{lv,k} - g_{kl,v})_m\right] =$$

$$= \frac{1}{2}\delta^v_i\left(g_{kv,m,l} + g_{mv,k,l} - g_{km,v,l} - g_{kv,l,m} - g_{lv,k,m} + g_{kl,v,m}\right).$$

Taking into account that

$$g_{kv,m,l} = g_{kv,l,m} \ \text{ and } \ g_{ik} = g_{ki},$$

we have

$$R_{iklm} = \frac{1}{2}(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}). \tag{H.5}$$

To calculate R_{kilm} let us produce the following permutation in (H.5): $i \leftrightarrow k$;then

$$R_{kilm} = \frac{1}{2}(g_{km,i,l} + g_{il,k,m} - g_{kl,i,m} - g_{im,k,l}) = -\frac{1}{2}(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}) = -R_{iklm}.$$

To calculate R_{ikml} let us produce the following permutation in (H.5): $l \leftrightarrow m$;then

$$R_{ikml} = \frac{1}{2}(g_{il,k,m} + g_{km,i,l} - g_{im,k,l} - g_{kl,i,m}) = -\frac{1}{2}(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}) = -R_{iklm}.$$

b) *Permutation of pairs*

$$R_{iklm} = R_{lmik}.$$

Solution: Producing the permutation $(ik) \leftrightarrow (lm)$ in (H.5), we have

$$R_{lmik} = \frac{1}{2}(g_{lk,m,i} + g_{mi,l,k} - g_{li,m,k} - g_{mk,l,i}) = \frac{1}{2}(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}) = R_{iklm}.$$

Q2. *25 Marks*

a) Prove that

$$R_{iklm} + R_{imkl} + R_{ilmk} = 0.$$

Solution: With the help of circling permutation

$$\begin{array}{ccc} & \rightarrow & l \\ k & & \downarrow \\ & \leftarrow & m \end{array}$$

we have

$$R_{iklm} + R_{imkl} + R_{ilmk} =$$

$$= \frac{1}{2}[(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}) + (g_{ik,l,m} + g_{lm,i,k} - g_{im,l,k} - g_{lk,i,m}) + (g_{il,m,k} + g_{mk,i,l} - g_{ik,m,l} - g_{ml,i,k})] =$$

$$= \frac{1}{2}[(g_{im,k,l} - g_{im,l,k}) + (g_{kl,i,m} - g_{lk,i,m}) - (g_{il,k,m} - g_{il,m,k}) - (g_{km,i,l} - g_{mk,i,l}) + (g_{ik,l,m} - g_{ik,m,l}) + (g_{lm,i,k}) - g_{ml,i,k})] = 0.$$

b) Using a locally-inertial coordinate system prove the Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Solution: In a locally-inertial frame of reference

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m}$$

and

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k}.$$

With the same circling permutation of indices as before

$$\begin{array}{ccc} & \rightarrow & l \\ k & & \downarrow \\ & \leftarrow & m \end{array}$$

we obtain

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} =$$

$$= (\Gamma^n_{il,k} - \Gamma^n_{ik,l})_{,m} + (\Gamma^n_{im,l} - \Gamma^n_{il,m})_{,k} + (\Gamma^n_{ik,m} - \Gamma^n_{im,k})_{,l} =$$

$$= \Gamma^n_{il,k,m} - \Gamma^n_{ik,l,m} + \Gamma^n_{im,l,k} - \Gamma^n_{il,m,k} + \Gamma^n_{ik,m,l} - \Gamma^n_{im,k,l} =$$

$$= (\Gamma^n_{il,k,m} - \Gamma^n_{il,m,k}) - (\Gamma^n_{ik,l,m} - \Gamma^n_{ik,m,l}) + (\Gamma^n_{im,l,k} - \Gamma^n_{im,k,l}) = 0.$$

Q3. *25 Marks*

a) Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

Solution:

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk}.$$

Producing summation in the expression (H.1) for the Riemann tensor $i = l$ and replacing k by i and m by k we have

$$R_{ik} = R^l_{ilk} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{nl}\Gamma^n_{ik} - \Gamma^l_{nk}\Gamma^n_{il} =$$

$$= \Gamma^l_{ik,l} - \Gamma^l_{il,k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

b) *Using the Bianchi identity, prove that the Ricci tensor and the scalar curvature $R = g^{ik}R_{ik}$ satisfy the following identity:*

$$R^l_{m;l} = \frac{1}{2}\frac{\partial R}{\partial x^m}.$$

Solution: After contracting the Bianchi identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0.$$

According to the definition of Ricci tensor

$$R^i_{kil} = R_{kl},$$

the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}.$$

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km},$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}.$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip}R_{pklm;i},$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip}R_{pklm;i} = -g^{ip}R_{kplm;i} = g^{ip}R_{kpml;i}.$$

After all these manipulations we have

$$g^{ip}R_{kpml;i} + R_{kl;m} - R_{km;l} = 0.$$

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km}g^{ip}R_{kpml;i} + g^{km}R_{kl;m} - g^{km}R_{km;l} = \left(g^{km}g^{ip}R_{kpml}\right)_{;i} + \left(g^{km}R_{kl}\right)_{;m} - \left(g^{km}R_{km}\right)_{;l} = 0.$$

In the first term expression in brackets can be simplified as

$$g^{km}g^{ip}R_{kpml} = g^{ip}R_{pl} = R^i_l.$$

In the second term expression in brackets can be simplified as

$$g^{km}R_{kl} = R^m_l.$$

According to the definition of scalar curvature

$$R = g^{km}R_{km},$$

the third term can be simplified as

$$\left(g^{km}R_{km}\right)_{;l} = R_{;l} = R_{,l}.$$

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0,$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0,$$

finally

$$R^i_{l;i} - \frac{1}{2}R_{,l} = 0. \tag{H.6}$$

Q4. *25 Marks*

a) Starting from the Einstein equations in the form

$$R_{ik} = \frac{8\pi G}{c^4}\left(T_{ik} - \frac{1}{2}g_{ik}T\right),$$

prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right),$$

where δ^i_k is the unit diagonal four-tensor.

Solution: Let us first rewrite the EFES in the mixed form

$$R^i_n = g^{ik}R_{kn} = \frac{8\pi G}{c^4}\left(g^{ik}T_{kn} - \frac{1}{2}g^{ik}g_{kn}T\right) = \frac{8\pi G}{c^4}\left(T^i_n - \frac{1}{2}\delta^i_n T\right).$$

Then by summation $i = n$ we obtain that

$$R = R^n_n = \frac{8\pi G}{c^4}\left(T^n_n - \frac{1}{2}\delta^n_n T\right) = \frac{8\pi G}{c^4}\left(T - \frac{1}{2}4T\right) = \frac{8\pi G}{c^4}\left(T - 2T\right) = -\frac{8\pi G}{c^4}T,$$

hence

$$T^i_k = \frac{c^4}{8\pi G}R^i_k + \frac{1}{2}\delta^i_k T = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

b) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

Solution:$R_{ik} = 0$ implies that $T_{ik} = 0$, i.e. the space-time is empty. $R_{ikln} \neq 0$ means that the space-time is not flat. [Thus we have free gravitational field in empty space-time which is gravitational wave.]

c) Prove that the energy-momentum tensor of matter T^i_k satisfies the conservation law $T^k_{i;k} = 0$.

Solution: Taking covariant divergence of the EFEs in mixed form we obtain

$$R^i_{k;i} - \frac{1}{2}\delta^i_k R_{,i} = \frac{8\pi G}{c^4}T^i_{k;i}. \tag{H.7}$$

As follows from (H.6)(see Q3b) the LHS of (H.7) is equal to zero, hence

$$T^i_{k;i} = 0.$$