F. Solutions to course work 3

Q1. *25 Marks*

a)Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity.

Solution: A uniform gravitational field is equivalent to a uniform acceleration.

b) Explain the similarity between an "actual" gravitational field and a non-inertial reference system. Give the definition of a locally Galilean coordinate system.

Solution: The fundamental property of gravitational fields that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence": $m_{in} = m_g$. The GR gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field.

c) Explain why an "actual" gravitational field cannot be eliminated by any transformation of coordinates over all space-time.

Solution: Globally (not locally), "actual" Gravitational Fields can be distinguished from corresponding non-inertial frame of reference by its behavior at infinity: Gravitational fields generated by gravitating bodies fall with distance.

d) Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

Solution: The shape of all physical equations should be the same in an arbitrary frame of reference. This principle is a mathematical formulation of the Principle of Equivalence.

Q2. *25 Marks*

a) Give the definition of a contravariant vector in terms of the transformation of curvilinear coordinates.

Solution: The Contravariant four-vector is the combination of four quantities (components) A^i, which are transformed like differentials of coordinates:

$$A^i = S^i_k A'^k, \text{ where } S^i_k \frac{\partial x^i}{\partial x'^k}. \tag{F.1}$$

b) Give the definition of a covariant vector in terms of the transformation of curvilinear coordinates.

Solution: The Covariant four-vector is the combination of four quantities (components) A_i, which are transformed like like components of the gradient of a scalar field:

$$A_i = \frac{\partial x'^k}{\partial x^i} A'_k, \text{ where } \tilde{S}^i_k = \frac{\partial x'^i}{\partial x^k}. \tag{F.2}$$

c) What is the mixed tensor of the second rank in terms of the transformation of curvilinear coordinates (you can assume that a mixed tensor of the second rank is transformed as a product of covariant and contrvariant vectors).

Solution: Mixed tensor of the 2 rank has $4^2 = 16$ components and 2 indices, 1 contravariant and 1 covariant. Corresponding transformation law is the same as for a product $B^i C_k = (S^i_n B^n)(\tilde{S}^m_k C_m)$, hence

$$A^i_k = S^i_n \tilde{S}^m_k A'^n_m, \tag{F.3}$$

we see 2 transformation matrices in the transformation law.

d) Explain why the principle of covariance implies that all physical equations should contain only tensors.

Solution: By definition, tensors are objects which are transformed properly in the course of coordinate transformations from one frame of reference to another keeping the shape of any physical equation unchanged as it is required by the covariance principle.

Q3. *25 Marks*

a) Prove that the metric tensor is symmetric. Give a rigorous proof that the interval is a scalar.

Solution:

$$ds^2 = g_{ik} dx^i dx^k = \frac{1}{2}(g_{ik} dx^i dx^k + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} dx^k dx^i + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik}) dx^i dx^k =$$

$$= \tilde{g}_{ik} dx^i dx^k, \tag{F.4}$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}), \tag{F.5}$$

which is obviously a symmetric one. Then we just drop "~"(The end of proof).

$$ds^2 = g_{ik}dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm}dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm}dx'^p dx'^w) = g'_{pw}dx'^p dx'^w = g'_{ik}dx'^i dx'^k = ds'^2, \tag{F.6}$$

hence $ds = ds'$ which means that ds is a scalar.(The end of proof).

b) Give the definition of the reciprocal tensors of the second rank. What is the contravariant metric tensor g^{ik}.

Solution: Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta_i^l. \tag{F.7}$$

The contravariant metric tensor g^{ik} is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta_i^l. \tag{F.8}$$

Solution Q3(b)

c) Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

Solution: We know that in the galilean frame of reference

$$g^{ik} = \eta^{ik}, \tag{F.9}$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}. \tag{F.10}$$

d) Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa.

Solution:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k, \tag{F.11}$$

we can rise and descend indices as we like.

e) Show with the help of straightforward differentiation that if A^i is a vector then dA^i is not a vector.

Solution:

$$A_i = \frac{\partial x'^k}{\partial x^i}A'_k \;\; dA_i = \frac{\partial x'^k}{\partial x^i}dA'_k + A'_k\frac{\partial^2 x'^k}{\partial x^i \partial x^l}dx^l, \tag{F.12}$$

Q4.*25 Marks*

a) Motivate the necessity to introduce parallel translation of a vector. Explain the meaning of the Christoffel symbols. Explain why the Christoffel symbols do not form a tensor.

Solution: In arbitrary coordinates to obtain a differential of a vector which forms a vector we should subtract vectors in the same point, not in different as we have done before. Hence we need produce a parallel transport or a parallel translation.

Under a parallel translation of a vector in galilean frame of reference its component don't change, but in curvilinear coordinates they do and we should introduce some corrections:

$$DA^i = dA^i - \delta A^i. \tag{F.13}$$

These corrections obviously should be linear with respect to all components of A_i and independently they should be linear with respect of dx^k, hence we can write these corrections as

$$\delta A^i = -\Gamma^i_{kl} A^k dx^l, \tag{F.14}$$

where Γ^i_{kl} are called Christoffel Symbols which obviously don't form any tensor, because DA_i is the tensor while as we know dA_i is not a tensor.

b) Show that

$$\Gamma^i_{km} = \frac{1}{2} g^{in} \left(g_{kn,m} + g_{mn,k} - g_{km,n} \right).$$

Solution: We know that

$$\Gamma^l_{ik} = \Gamma^l_{ki} \tag{F.15}$$

and

$$g_{ik;m} = 0. \tag{F.16}$$

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km} \Gamma^m_{il}, \tag{F.17}$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk} \Gamma^m_{il} - g_{im} \Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0. \tag{F.18}$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i, \tag{F.19}$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \text{ and } -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}. \tag{F.20}$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li}, \tag{F.21}$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\, kl} \tag{F.22}$$

Finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right). \tag{F.23}$$

c) Explain why for the derivation of physical equations in the presence of a gravitational field one can simply replace partial derivatives by covariant derivatives, and as an example show that the motion of a particle in a gravitational field is given by the geodesic equation

$$\frac{d^2 x^i}{ds^2} + \Gamma^i_{kl} \frac{dx^k}{ds} \frac{dx^l}{ds} = 0.$$

Solution: Gravity is equivalent to curved space-time, hence in all differentials of tensors we should take into account the change in the components of a tensor under an infinitesimal parallel transport. Corresponding corrections are expressed in terms of the Cristoffel symbols and reduced to replacement of any partial derivative by corresponding covariant derivative.

$$u^i = \frac{dx^i}{ds} \tag{F.24}$$

is the four-velocity. The equation for motion of a free particle in absence of gravitational field is

$$\frac{du^i}{ds} = 0. \tag{F.25}$$

In presence of a gravitational field this equation is generalized to the equation

$$\frac{Du^i}{ds} = 0, \tag{F.26}$$

which gives

$$\frac{Du^i}{ds} = \frac{du^i}{ds} + \Gamma^i_{kn} u^k \frac{dx^n}{ds} = \frac{d^2 x^i}{ds^2} + \Gamma^i_{kn} u^k u^n = 0. \tag{F.27}$$