Queen Mary
University of London

M. Sci. Examination by course unit 2010

MTHM033/MTH720U Relativity and gravitation

Duration: 3 hours

Date and time: xx xxx 2010, xxxxh

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The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

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Complete all rough workings in the answer book and cross through any work which is not to be assessed.

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Exam papers must not be removed from the examination room.

Examiner(s): Polnarev

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ ***N m***^{2} ***kg***$^{-2}$
Speed of light	c	$= 3 \times 10^8$ ***m s***$^{-1}$
1 kpc		$= 3 \times 10^{19}$ ***m***

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ ***and take on the values 1, 2, 3.***
Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ ***and take on the values 0, 1, 2, 3.***
The metric signature $(+ - - -)$ ***is used.***
Partial derivatives are denoted by ",".
Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m,$ ***where*** Γ^i_{kn} ***are Christoffel symbols***

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ ***is the 4-velocity along the geodesic***.

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \text{ ***where*** } R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein tensor:

$$G_{ik} = R_{ik} - 1/2 g_{ik} R.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{(1 - \frac{r_g}{r})} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \textbf{\textit{km}},\ \ \textbf{\textit{where}}\ M_\odot\ \textbf{\textit{is the mass of Sun}}.$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2 r_g r a c}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2 \delta_{\alpha\beta}) dM$$

is the quadrupole tensor.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 Prove that the metric tensor is symmetric. Explain how this symmetry and dimensions of space-time pre-determine the total number of the Einstein Field Equations required for the description of space-time geometry.

Question 2 Give the definition of the contravariant metric tensor g^{ik}. What manipulations with indices can be produced with the help of g_{ik} and g^{ik}? Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0} S^k_{(0)0} - S^i_{(0)1} S^k_{(0)1} - S^i_{(0)2} S^k_{(0)2} - S^i_{(0)3} S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from a locally inertial frame of reference (local Galilean frame) to this non-inertial frame.

Question 3 Transformation from a local inertial (or local Galilean) frame of reference $x^i_{(0)}$ to some non-inertial frame x^i is given by the following transformation matrix: $S^i_{(0)k} = \delta^i_k + f\delta^i_0\delta^0_k$, where $f = f(x^m_{(0)})$ is a scalar field. Using the result of Question 1, show that the metric in the non-inertial frame of reference x^i has the following form: $ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.

Question 4 Explain why in order to prove that some tensor is identically equal to zero it is enough to show that all components of this tensor are equal to zero in the local galilean frame of reference. Then, prove that the Christoffel symbols, Γ^i_{kl}, are symmetric with respect to their low indices.

Question 5 Prove that all covariant derivatives of the metric tensor are equal to zero, - i.e., $g_{ik;l} = 0$. Then, using the symmetry of the Christoffel symbols proofed in question 4, show that the Christoffel symbols in terms of the metric tensor are $\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m}\right)$.

Question 6 Prove that the determinant of the metric tensor, g, is negative in all frames of reference. Then, prove the following identity:

$$2d \ln \sqrt{-g} = g^{ik} dg_{ik} = -g_{ik} dg^{ik}.$$

Question 7 Consider a light ray (electromagnetic signal) propagating in a gravitational field. The four-dimensional wave vector for the electromagnetic signal is defined as $k^i = dx^i/d\lambda$, where λ is some parameter varying along the ray. The scalar function Ψ is called the eikonal and defined as $k_i = \Psi_{,i}$. Derive the Eikonal equation (i.e., the equation for Ψ) and explain how using this equation one can describe the propagation of electromagnetic signals in a given gravitational field.

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 .

(a) Prove the following identities:

$$\Gamma^i_{ik} - (\ln \sqrt{-g})_{,k} = 0 \text{ and } [\sqrt{-g} g^{ik})]_{,k} + \sqrt{-g} g^{kl} \Gamma^i_{kl} = 0.$$

[13]

(b) Prove that the covariant divergence of an arbitrary contravariant vector can be written as

$$A^i_{;i} = \frac{1}{\sqrt{-g}} (\sqrt{-g} A^i)_{,i}.$$

Show that the analogous expression can be written for an antisymmetric tensor of the second rank A^{ik}:

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}} (\sqrt{-g} A^{ki})_{,i}.$$

[9]

Question 9 .

(a) Give brief explanation of what is meant by the limit of stationarity and the event horizon of a black hole and how to determine their locations. What is meant by ergosphere and where it is located? [11]

(b) Consider a rotating black hole described by the Kerr metric given in the rubric. Find the mass (express your result in solar masses) and angular momentum parameter of the black hole, $\alpha = 2a/r_g$, if its ergosphere in the equatorial plane ($\theta = \pi/2$) lies between $r_{min} = 125$km and $r_{max} = 150$km. [11]

Question 10 .

(a) Prove the Bianchi identity. [8]

(b) Prove that the covariant Riemann tensor $R_{iklm} = g_{in} R^n_{klm}$ is antisymmetric in each of the index pairs i,k and l,m ($R_{iklm} = -R_{kilm} = -R_{ikml}$) and is symmetric under the interchange of two pairs with one another ($R_{iklm} = R_{lmik}$). Using these properties, show that by contracting the Bianchi identity on the pairs of indices i,k and l,n, one obtains that the covariant divergence of the Einstein tensor G^i_k (see rubric) is equal to zero. [14]

Question 11 .

(a) A weak gravitational wave is a small perturbation of the Minkowski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that, to terms of first order in h_{ik}, the contravariant metric tensor is $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Consider a linear transformation $x^i = x'^i + \xi^i$, where ξ^i are small functions of x^i. Show that $h_{ik} = h'_{ik} - \xi_{i,k} - \xi_{k,i}$. Prove that it is always possible to find such ξ^i that the Ricci tensor takes the following simple form:

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[14]

(b) Two bodies of equal mass, $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves, show that in order of magnitude, $h \sim (r_g/R)(r_g/cP)^{2/3}$, where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius. [8]

End of Paper

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Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 *Prove that the metric tensor is symmetric. Explain how this symmetry and dimensions of space-time pre-determine the total number of the Einstein Field Equations required for the description of space-time geometry.*

Solution 1 *[Seen similar]*

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ik}dx^i dx^k}{2} = \frac{g_{ik}dx^i dx^k + g_{ik}dx^k dx^i}{2}.$$

The following substitution in the second term

$$i \to k \ \ k \to i$$

gives

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ki}dx^i dx^k}{2} = \frac{g_{ik} + g_{ki}}{2} dx^i dx^k = \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{g_{ik} + g_{ki}}{2}.$$

[4]

Obviously that

$$\tilde{g}_{ik} = \tilde{g}_{ki}.$$

We can use $\tilde{g}_{ik}$ instead g_{ik} and then changing notations just drop $\tilde{}$. [2]

Space-time is four dimensional, i.e g_{ik} have 4×4 component. Due to the symmetry there only $4+3+2+1 = 10$ independent components. Hence, to describe geometry of 4-space time one needs 10 equations. [2]

Question 2 *Give the definition of the contravariant metric tensor* g^{ik}*. What manipulations with indices can be produced with the help of* g_{ik} *and* g^{ik}*? Show that in an arbitrary non-inertial frame*

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ *is the transformation matrix from a locally inertial frame of reference (local Galilean frame) to this non-inertial frame.*

Solution 2 *[Seen similar]*

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta^l_i.$$

[1]

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta^l_i.$$

[1]

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

[1]

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1).$$

[2]

Hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

[3]

Question 3 *Transformation from a local inertial (or local Galilean) frame of reference $x^i_{(0)}$ to some non-inertial frame x^i is given by the following transformation matrix: $S^i_{(0)k} = \delta^i_k + f\delta^i_0\delta^0_k$, where $f = f(x^m_{(0)})$ is a scalar field. Using the result of Question 1, show that the metric in the non-inertial frame of reference x^i has the following form: $ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2$.*

Solution 3 *[Seen similar]*

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{nm} = (\delta^i_n + f\delta^i_0\delta^0_n)(\delta^k_m + f\delta^k_0\delta^0_m)\eta^{nm} =$$

$$= [\delta^i_n\delta^k_m + f(\delta^i_n\delta^k_0\delta^0_m + \delta^k_m\delta^i_0\delta^0_n) + f^2\delta^i_0\delta^0_n\delta^k_0\delta^0_m]\eta^{nm} =$$

$$= \eta^{ik} + f(\eta^{i0}\delta^k_0 + \eta^{0k}\delta^i_0) + f^2\eta^{00}\delta^i_0\delta^k_0 = \eta^{ik} + 2f\delta^i_0\delta^k_0 + f^2\delta^i_0\delta^k_0 =$$

$$= \begin{pmatrix} (1+f)^2 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[5]

Determinant $|g^{ik}| = -(1+f)^2$, hence g_{ik} which is reciprocal to g^{ik}, is presented by inverse matrix:

$$g_{ik} = \begin{pmatrix} (1+f)^{-2} & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[2]

Finally

$$ds^2 = (1+f)^{-2}(dx^0)^2 - (dx^1)^2 - (dx^2)^2 - (dx^3)^2.$$

[1]

Question 4 *Explain why in order to prove that some tensor is identically equal to zero it is enough to show that all components of this tensor are equal to zero in the local galilean frame of reference. Then, prove that the Christoffel symbols,* Γ^i_{kl}, *are symmetric with respect to their low indices.*

Solution 4. *[Seen similar]*

Transformation of tensors from the local galilean frame of reference to an arbitrary frame of reference is produced with the help of matrices S^i_k and $\tilde{S}^i_k$. It does not matter how many times these matrices appear in the transformation law, the resulting components of the tensor in a new frame of references are linear combinations of the components in the local galilean frame of reference, hence all new components are automatically equal to zero in an arbitrary frame of reference, if they are zero in the local galilean frame of reference. [2]

Let $A_i = \phi_{,i}$, where ϕ is a scalar, then

$$A_{i;k} - A_{k;i} = A_{i,k} - \Gamma^m_{ik} A_m - A_{k,i} + \Gamma^m_{ki} A_m = \phi_{,i,k} - \phi_{,k,i} + (\Gamma^m_{ki} - \Gamma^m_{ik}) A_m = (\Gamma^m_{ki} - \Gamma^m_{ik}) A_m.$$

[3]

LHS is a tensor. In a local galilean coordinates RHS=0, hence in the local galilean coordinates LHS=0. Thus LHS=0 in all coordinates, and finally taking into account that A_m is an arbitrary vector, we conclude that $\Gamma^m_{ki} - \Gamma^m_{ik} = 0$ in all coordinates, hence $\Gamma^m_{ki} = \Gamma^m_{ik}$. [3]

Question 5 *Prove that all covariant derivatives of the metric tensor are equal to zero, - i.e.,* $g_{ik;l} = 0$. *Then, using the symmetry of the Christoffel symbols proofed in question 4, show that the Christoffel symbols in terms of the metric tensor are* $\Gamma^i_{kl} = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m})$.

Solution 5. *[Seen similar]*

Let A_i is an arbitrary covariant vector. By the definition of D one can say that DA_i is also vector and its contravariant representation is

$$DA^i = g^{ik} DA_k.$$

[1]

On other hand

$$DA^i = D(g^{ik} A_k) = Dg^{ik} A_k + g^{ik} DA_k,$$

hence

$$g^{ik} DA_k = Dg^{ik} A_k + g^{ik} DA_k$$

which means that

$$Dg^{ik} A_k = 0$$

for arbitrary vector A_k, hence

$$Dg^{ik} = 0.$$

[2]

This means that

$$Dg_{ik} = g_{ik;l} dx^l = 0,$$

for arbitrary dx^l, which means that all $g_{ik;l} = 0$. [1]

We can apply covariant differentiation to g_{ik}:

$$g_{ik;l} = g_{ik,l} - \Gamma^m_{il} g_{mk} - \Gamma^m_{kl} g_{mi} = 0,$$

or after two cycling permutations of indices $i \to k$ $k \to l$ $l \to i$ we have

$$g_{ik,l} = \Gamma^m_{il} g_{mk} + \Gamma^m_{kl} g_{mi},$$

$$g_{kl,i} = \Gamma^m_{ki} g_{mk} + \Gamma^m_{li} g_{mk},$$

$$-g_{li,k} = -\Gamma^m_{lk} g_{ml} - \Gamma^m_{ik} g_{ml}.$$

[3]

Taking into account that $\Gamma^m_{ki} = \Gamma^m_{ik}$ and $g_{ik} = g_{ki}$we obtain by summation of LHSs and RHSs that

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m}).$$

[1]

Question 6 *Prove that the determinant of the metric tensor, g, is negative in all frames of reference. Then, prove the following identity:*

$$2d\ln\sqrt{-g} = g^{ik}dg_{ik} = -g_{ik}dg^{ik}.$$

Solution 6 *[Seen similar]*

Taking into account that g_{ik} and g^{ik} are reciprocal, one obtains that $g \equiv det(g_{ik}) = 1/det(g^{ik})$. [1]

We know (see question 3) that

$$g^{ik} = S^i_{(0)n} S^k_{(0)m} \eta^{lm}.$$

Obviously, $det(\eta^{lm} = -1$, hence

$$det(g^{ik}) = detS^i_{(0)n} \times detS^k_{(0)m} \times det(\eta^{lm}) = -S^2,$$

where S is the determinant of the transformation matrix. One can see that $g = -S^{-2} < 0$ in all frames of reference. [2]

The determinant g depends on all components g_{ik}. Calculating g with the help, say the first raw, one can write

$$g = M^{1i} g_{1i},$$

where M^{1i} are minors of the components in the first row. Obviously M^{1i} do not contain g_{1i}. Hence

$$\frac{\partial g}{\partial g_{1i}} = M^{1i}.$$

This is true for any row in determinant, thus

$$\frac{\partial g}{\partial g_{ni}} = M^{ni}.$$

[2]

Taking into account that g^{ik} is inverse matrix of g_{ik}, one can write $g^{ik} = M^{ik}/g$, i.e. $M^{ik} = gg^{ik}$. Thus

$$dg = \frac{\partial g}{\partial g_{ik}} dg_{ik} = M^{ik} dg_{ik} = gg^{ik} dg_{ik},$$

hence

$$g^{ik}dg_{ik} = \frac{dg}{g} = d\ln|g| = d\ln(-g) = 2\ln\sqrt{-g}.$$

[2]

Then

$$g^{ik}dg_{ik} = d(g^{ik}g_{ik}) - g_{ik}dg^{ik} = d\delta^i_i - g_{ik}dg^{ik} = -g_{ik}dg^{ik}.$$

[1]

Question 7 *Consider a light ray (electromagnetic signal) propagating in a gravitational field. The four-dimensional wave vector for the electromagnetic signal is defined as $k^i = dx^i/d\lambda$, where λ is some parameter varying along the ray. The scalar function Ψ is called the eikonal and defined as $k_i = \Psi_{,i}$. Derive the Eikonal equation (i.e., the equation for Ψ) and explain how using this equation one can describe the propagation of electromagnetic signals in a given gravitational field.*

Solution 7. *[Seen similar]*

We know that along any light ray $ds^2 = g_i k dx^i dx^k = 0$. [1]

Thus we have

$$g_{ik}k^i k^k = g_{ik}\frac{dx^i}{d\lambda}\frac{dx^k}{d\lambda} = \frac{g_{ik}dx^i dx^k}{d\lambda^2} = \frac{ds^2}{d\lambda^2} = 0.$$

[2]

For an arbitrary covariant vector $k_i = g_{ik}k^k$ we can find such a scalar that $k_i = \Psi_{,i}$. [1]

Substituting this to the previous formula we obtain the following equation for Ψ:

$$g^{ik}S_{,i}S_{,k} = 0.$$

This equation is called the Eikonal equation. [2]

The Eikonal equation "works" in the following way:
(i) We solve this single equation for single scalar field $\Psi(x^m)$;
(ii) Taking partial derivatives we calculate covariant components of the four-dimensional wave vector $k_i = -\Psi_{,i}$;
(iii) With the help of g^{ik} we obtain contravariant components of the four-dimensional wave vector
(iv) Finally we calculate world lines of electromagnetic signals:

$$x^i(s) = \int k^i d\lambda.$$

[3]

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 (a) *Prove the following identities:*

$$\Gamma^i_{ik} - (\ln\sqrt{-g})_{,k} = 0 \ \ \textit{and} \ \ [\sqrt{-g}g^{ik})]_{,k} + \sqrt{-g}g^{kl}\Gamma^i_{kl} = 0.$$

[13]

Solution B1a *[Seen similar]*

$$\Gamma^i_{ik} = \frac{1}{2}g^{in}\left(g_{in,k} + g_{kn,i} - g_{ik,n}\right) = \frac{1}{2}g^{in}g_{in,k} + \frac{1}{2}g^{in}g_{kn,i} - \frac{1}{2}g^{in}g_{ik,n},$$

changing indices of summation in the last term, $i \to n$, $n \to i$, one obtains

$$\Gamma^i_{ik} = \frac{1}{2}g^{in}g_{in,k} + \frac{1}{2}g^{in}g_{kn,i} - \frac{1}{2}g^{ni}g_{nk,i} = \frac{1}{2}g^{in}g_{in,k} = (\ln\sqrt{-g})$$

(see question 6). Hence

$$\Gamma^i_{ik} - (\ln\sqrt{-g})_{,k} = 0.$$

[4]

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2}g^{kl}g^{in}\left(g_{kn,l} + g_{ln,k} - g_{kl,n}\right) = \frac{1}{2}g^{kl}g^{in}g_{kn,l} + \frac{1}{2}g^{kl}g^{in}g_{ln,k} - \frac{1}{2}g^{kl}g^{in}g_{kl,n},$$

changing indices of summation in the second term, $k \to l$, $l \to k$, one obtains

$$g^{kl}\Gamma^i_{kl} = \frac{1}{2}g^{kl}g^{in}g_{kn,l} + \frac{1}{2}g^{lk}g^{in}g_{kn,l} - \frac{1}{2}g^{kl}g^{in}g_{kl,n} = g^{kl}g^{in}g_{kn,l} - \frac{1}{2}g^{in}\frac{dg}{g}$$

(see question 6). [4]

Then taking into account that

$$\frac{(\sqrt{-g})_{,n}}{\sqrt{-g}} = \frac{1}{2}\frac{-g_{,n}}{\sqrt{-g}\sqrt{-g}} = \frac{1}{2}\frac{-g_{,n}}{-g} = \frac{g_{,n}}{2g},$$

one obtains

$$g^{kl}\Gamma^i_{kl} = g^{in}\left(g^{kl}g_{kn,l} - \frac{(\sqrt{-g})_{,n}}{\sqrt{-g}}\right) = \frac{g^{in}}{\sqrt{-g}}\left(\sqrt{-g}g^{kl}g_{kn,l} - (\sqrt{-g})_{,n}\right) =$$

$$= \frac{g^{in}}{\sqrt{-g}}\left[\sqrt{-g}(g^{kl}g_{kn})_{,l} - g_{kn}g^{kl}_{,l} - (\sqrt{-g})_{,n}\right] = \frac{1}{\sqrt{-g}}\left[\sqrt{-g}(g^{in}\delta^k_n)_{,l} - g^{in}g_{kn}g^{kl}_{,l} - g^{in}(\sqrt{-g})_{,n}\right] =$$

$$= \frac{1}{\sqrt{-g}}\left[-\sqrt{-g}\delta^i_k g^{kl}_{,l} - g^{in}(\sqrt{-g})_{,n}\right] = \frac{1}{\sqrt{-g}}\left[-\sqrt{-g}g^{in}_{,n} - g^{in}(\sqrt{-g})_{,n}\right] = -\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{in}\right)_{,n},$$

hence

$$[\sqrt{-g}g^{ik})]_{,k} + \sqrt{-g}g^{kl}\Gamma^i_{kl} = 0.$$

[5]

(b) *Prove that the covariant divergence of an arbitrary contravariant vector can be written as*

$$A^i_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}.$$

Show that the analogous expression can be written for an antisymmetric tensor of the second rank A^{ik}:

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{ki})_{,i}.$$

[9]

Solution B1b. *[Seen similar]*

$$A^i_{;i} = A^i_{,i} + \Gamma^i_{in}A^n = A^i_{,i} + (\ln\sqrt{-g})_{,n}A^n$$

(see the previous sub-question). Taking into account that

$$(\ln\sqrt{-g})_{,n} = \frac{(\sqrt{-g})_{,n}}{(\sqrt{-g})},$$

one obtains

$$A^i_{;i} = A^i_{,i} + \frac{(\sqrt{-g})_{,n}}{(\sqrt{-g})}A^n = \frac{1}{\sqrt{-g}}\left(\sqrt{-g}A^i_{,i} + (\sqrt{-g})_{,i}A^i\right) = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^i)_{,i}.$$

[4]

$$A^{ki}_{;i} = A^{ki}_{,i} + \Gamma^k_{in}A^{ni} + \Gamma^i_{in}A^{kn}.$$

Since $A^{ni} = -A^{in}$

$$\Gamma^k_{in}A^{ni} = -\Gamma^k_{in}A^{in} = -\Gamma^k_{ni}A^{in} = -\Gamma^k_{in}A^{ni}, \text{ hence } \Gamma^k_{in}A^{ni} = 0.$$

Thus

$$A^{ki}_{;i} = A^{ki}_{,i} + (\ln\sqrt{-g})_{,i}A^{ki}$$

(see the previous sub-question) and finally

$$A^{ki}_{;i} = \frac{1}{\sqrt{-g}}(\sqrt{-g}A^{ki})_{,i}.$$

[5]

Question 9 (a) *Give brief explanation of what is meant by the limit of stationarity and the event horizon of a black hole and how to determine their locations. What is meant by ergosphere and where it is located?* [11]

Solution B2a. *[seen similar]*

The surface $g_{00} = 0$ (this equation determines the location of this surface) is called the limit of stationarity. No particle can be in rest inside this surface [but it does not mean that such a particle should move inward.]. Let us consider ds for the test particle in rest, i.e. put $dr = d\theta = d\phi = 0$, in this case

$$ds^2 = g_{00} dx^{0^2}.$$

If $g_{00} = 0$ then $ds^2 = 0$, which means that the world line of the particle at rest is the world line of light, hence at the surface $g_{00} = 0$ no particle with finite rest mass can be at rest. [4]

The surface $g^{11} = 0$ (this equation determines the location of this surface)is called the event horizon. No particle can move outward from inside this surface. Let us consider a surface $F(r) = const$ and let $n_i = F_{,i}$ is its normal. If $g^{11} = 0$ then $g^{ik} n_i n_k = g^{11} n_1 n_1 = 0$, which means that n_i is the null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon [within the event horizon all particles should move inward.] [5]

The ergosphere is the region outside the event horizon, where rotational energy of the black hole is located, that is why it is possible to extract the rotational energy of the Kerr black hole.The ergosphere is located between the limit of stationarity and the event horizon. [2]

(b) *Consider a rotating black hole described by the Kerr metric given in the rubric. Find the mass (express your result in solar masses) and angular momentum parameter of the black hole, $\alpha = 2a/r_g$, if its ergosphere in the equatorial plane ($\theta = \pi/2$) lies between $r_{min} = 125km$ and $r_{max} = 150km$.* [11]

Solution B2b. *[seen similar]*

Location of the event horizon corresponds to

$$g^{11} = 0.$$

Taking into account that all out of diagonal components $g_{1i} = 0$ (if $i \neq i$), one can see that $g^{11} = 1/g_{11}$ and the location of event horizon can be determined from $g_{11} = \infty$ or, as follows from the expressions for Kerr metric given in the rubric, from

$$\Delta = r^2 - r_g r + a^2 = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2}}{2}.$$

The outer event horizon,r_{hor} corresponds to the sign "+", hence

$$r_{hor} = \frac{r_g}{2}\left(1 + \sqrt{1-\alpha^2}\right).$$

[4]

The location of limit of stationarity corresponds to

$$g_{00} = 0.$$

In the case of Kerr metric this corresponds to

$$1 - \frac{r_g r}{\rho^2} = 0,$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta,$$

hence from

$$r^2 - r_g r + a^2\cos^2\theta = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2\cos^2\theta}}{2}.$$

The outer limit of stationarity, r_{st} corresponds to the sign "+", hence

$$r_{st} = \frac{r_g}{2}\left(1 + \sqrt{1-\alpha^2\cos^2\theta}\right) = r_g = \frac{2GM}{c^2}$$

(because for the equatorial plane $theta = \pi/2$).Thus we have

$$r_{min} = \frac{GM}{c^2}\left(1 + \sqrt{1-\alpha^2}\right) \text{ and } r_{max} = \frac{2GM}{c^2}.$$

[4]

Hence $M/M_\odot = r_{max}/3\text{km} = 50$ and

$$\alpha = \sqrt{1 - \left(\frac{2r_{min}}{r_{max}} - 1\right)^2} = \frac{2}{r_{max}}\sqrt{r_{min}\left(r_{max} - r_{min}\right)} = \frac{2}{150}\sqrt{125 \times 25} = \frac{\sqrt{5}}{3} \approx 0.75.$$

[3]

Question 10 (a) *Prove the Bianchi identity.* [8]

Solution B3a. *[seen similar]*

The Bianchi identity, $R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0$, in the local galilean frame of reference, where all Christoffel symbols are equal to zero, can be re-written as

$$R^n_{ikl,m} + R^n_{imk,l} + R^n_{ilm,k} = 0$$

[2]

---

and the Riemann tensor in this frame can be written as

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl} = \Gamma^i_{km,l} - \Gamma^i_{kl,m},$$

[2]

$$R^n_{ikl,m}+R^n_{imk,l}+R^n_{ilm,k} = \left(\Gamma^n_{il,k} - \Gamma^n_{ik,l}\right)_{,m}+\left(\Gamma^n_{im,l} - \Gamma^n_{il,m}\right)_{,k}+\left(\Gamma^n_{ik,m} - \Gamma^n_{im,k}\right)_{,l} =$$

$$= \Gamma^n_{il,k,m} - \Gamma^n_{ik,l,m} + \Gamma^n_{im,l,k} - \Gamma^n_{il,m,k} + \Gamma^n_{ik,m,l} - \Gamma^n_{im,k,l} =$$

$$= [\Gamma^n_{il,k,m} - \Gamma^n_{il,m,k}] + [\Gamma^n_{ik,m,l} - \Gamma^n_{ik,l,m}] + [\Gamma^n_{im,l,k} - \Gamma^n_{im,k,l}] = [0] + [0] + [0] = 0.$$

[4]

(b) *Prove that the covariant Riemann tensor $R_{iklm} = g_{in}R^n_{klm}$ is antisymmetric in each of the index pairs i,k and l,m ($R_{iklm} = -R_{kilm} = -R_{ikml}$) and is symmetric under the interchange of two pairs with one another ($R_{iklm} = R_{lmik}$). Using these properties, show that by contracting the Bianchi identity on the pairs of indices i,k and l,n, one obtains that the covariant divergence of the Einstein tensor G^i_k (see rubric) is equal to zero.* [14]

Solution B3b. *[seen similar]*

In the local galilean frame of reference

$$R_{iklm} = \eta_{in}R^n_{klm} = \eta_{in}\left(\Gamma^n_{km,l} - \Gamma^n_{kl,m}\right) =$$

$$= \frac{1}{2}\eta_{in}\left[g^{np}(g_{kp,m} + g_{mp,k} - g_{km,p}\right]_{,l} - \frac{1}{2}\eta_{in}\left[g^{np}(g_{kp,l} + g_{lp,k} - g_{kl,p}\right]_{,m} =$$

$$= \frac{1}{2}\eta_{in}\eta^{np}\left(g_{kp,m,l} + g_{mp,k,l} - g_{km,p,l} - g_{kp,l,m} - g_{lp,k,m} + g_{kl,p,m}\right) =$$

$$= \frac{1}{2}\delta^p_i\left(g_{mp,k,l} - g_{km,p,l} - g_{lp,k,m} + g_{kl,p,m}\right) = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right).$$

[4]

$$R_{kilm} = \frac{1}{2}\left(g_{km,i,l} + g_{il,k,m} - g_{kl,i,m} - g_{im,k,l}\right) = -\frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= -R_{iklm}.$$

[2]

$$R_{ikml} = \frac{1}{2}\left(g_{il,k,m} + g_{km,i,l} - g_{im,k,l} - g_{kl,i,m}\right) = -\frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= -R_{iklm}.$$

[2]

$$R_{lmik} = \frac{1}{2}\left(g_{lk,m,i} + g_{mi,l,k} - g_{li,m,k} - g_{mk,l,i}\right) = \frac{1}{2}\left(g_{im,k,l} + g_{kl,i,m} - g_{il,k,m} - g_{km,i,l}\right) =$$
$$= R_{iklm}.$$

[2]

After contracting the Bianchi identity we obtain

$$g^{kl}R^i_{klm;i} + g^{kl}R^i_{kil;m} + g^{kl}R^i_{kmi;l} = g^{in}g^{kl}\left(R_{nklm;i} + R_{nkil;m} + R_{nkmi;l}\right) =$$
$$= g^{kl}g^{in}\left(-R_{knlm;i} + R_{nkil;m} - R_{nkim;l}\right) = -g^{in}R_{nm;i} + g^{kl}R_{kl;m} - g^{kl}R_{km;l} = -R^i_{m;i} + R_{,m} - R^l_{m;l} =$$
$$= -2R^i_{m;i} + R_{,m} = 0.$$

[3]

Hence
$$G^i_{k;i} = \left(R^i_k - \frac{1}{2}\delta^i_k R\right)_{;i} = -\frac{1}{2}\left(-2R^i_k + R_{,k}\right) = 0.$$

[1]

Question 11 (a) *A weak gravitational wave is a small perturbation of the Minkowski metric, $g_{ik} = \eta_{ik} + h_{ik}$. Show that, to terms of first order in h_{ik}, the contravariant metric tensor is $g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}$. Consider a linear transformation $x^i = x'^i + \xi^i$, where ξ^i are small functions of x^i. Show that $h_{ik} = h'_{ik} - \xi_{i,k} - \xi_{k,i}$. Prove that it is always possible to find such ξ^i that the Ricci tensor takes the following simple form:*

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[14]

Solution B4a. *[seen similar]*

If $g_{ik} = \eta_{ik} + h_{ik}$, where h_{ik} are small, contravariant metric tensor can be written as $g^{ik} = \eta^{ik} + a^{ik}$, where a^{ik} are also small. Taking into account that $g_{ik}g^{kn} = \delta_i^n$ we have

$$(\eta_{ik} + h_{ik})(\eta^{kn} + a^{kn}) = \delta_i^n, \quad \delta_i^n + \eta_{ik}a^{kn} + h_{ik}\eta^{kn} = \delta_i^n,$$

$$\eta_{ik}a^{kn} = -h_{ik}\eta^{kn}, \quad \eta^{im}\eta_{ik}a^{kn} = -\eta^{im}h_{ik}\eta^{kn}, \quad \delta_k^m a^{kn} = -\eta^{im}\eta^{kn}h_{ik},$$

$$a^{mn} = -\eta^{mi}\eta^{nk}h_{ik}, \quad \text{finally} \quad g^{ik} = \eta^{ik} - \eta^{in}\eta^{km}h_{nk}.$$

[3]

$$g_{ik} = \tilde{S}_i^n \tilde{S}_k^m g'_{nm}, \quad \text{where} \quad \tilde{S}_k^i = \frac{\partial x'^i}{\partial x^k} = \delta_k^i - \xi^i_{,k},$$

hence

$$\eta_{ik} + h_{ik} = (\delta_i^n - \xi^n_{,i})(\delta_k^m - \xi^m_{,k})(\eta_{nm} + h'_{nm});$$

to terms of first order in the h'_{ik}

$$h_{ik} = -\eta_{ik} + \delta_i^n[\delta_k^m(\eta_{nm} + h'_{nm}) - \xi^m_{,k}\eta_{nm}] - \xi^n_{,i}\delta_k^m\eta_{nm} = -\eta_{ik} + \eta_{ik} + h'_{ik} - \xi^m_{,k}\eta_{im} - \xi^n_{,i}\eta_{nk} =$$

$$= h'_{ik} - \xi_{i,k} - \xi_{k,i}.$$

[4]

Writing the Ricci tensors to terms of first order (in linear approximation) we have

$$R_{ik} = \Gamma^l_{ik,l} - \Gamma^l_{il,k} = \frac{1}{2}\eta^{lm}\left(h_{im,k,l} + h_{km,i,l} - h_{ik,m,l} - h_{im,l,k} - h_{lm,i,k} + h_{il,m,k}\right) =$$

$$= -\frac{1}{2}\eta^{lm}h_{ik,m,l} + \frac{1}{2}\eta^{lm}\left(h_{km,i,l} - h_{lm,i,k} + h_{il,m,k}\right) =$$

$$= -\frac{1}{2}\eta^{lm}h_{ik,m,l} + \frac{1}{2}\left(h^l_{k,i,l} - h_{,i,k} + h^m_{i,m,k}\right), \quad \text{where} \quad h = h^l_l.$$

[4]

We have four arbitrary functions ξ, thus we can impose on h_{ik} four supplementary conditions: $h^k_{i,k} - 1/2h_{,i} = 0$,

$$R_{ik} = -\frac{1}{2}\eta^{lm}h_{ik,l,m} + \frac{1}{2}\left(\frac{1}{2}h_{,k,i} - h_{,i,k} + \frac{1}{2}h_{,i,k}\right) = -\frac{1}{2}\eta^{lm}h_{ik,l,m}.$$

[3]

(b) *Two bodies of equal mass, $m_1 = m_2 = m$, attracting each other according to Newton's law, move in circular orbits around their common centre of mass with orbital period P. Using the quadrupole formula for the generation of gravitational waves, show that in order of magnitude, $h \sim (r_g/R)(r_g/cP)^{2/3}$, where R is the distance to the system and $r_g = \frac{2Gm}{c^2}$ is the gravitational radius.* [8]

Solution B4b. *[seen similar]*

To an order of magnitude (and omitting indices) we have

$$h \sim \frac{G}{c^4 R}\ddot{D} \sim \frac{G}{c^4 R} m r^2 P^{-2}.$$

[3]

Taking into account that according to Newton low

$$P^{-2} \sim Gmr^{-3}, \text{ we have } r \sim (GmP^2)^{1/3},$$

[2]

---

hence

$$h \sim \frac{Gm}{c^4 R P^2}\left(GmP^2\right)^{2/3} \sim \frac{r_g}{c^2 R P^2}\left(r_g c^2 P^2\right)^{2/3} \sim \frac{r_g}{R}\left(\frac{r_g}{cP}\right)^{2/3}.$$

[3]

End of Paper

QUEEN MARY,

UNIVERSITY OF LONDON

M. Sc. Examination 2009

MTHM033/MTH720U   Relativity and gravitation

Duration: 3 hours

Date and time: xx xxx 2009,   xxxxh

The paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Do not start reading the question paper until instructed to do so.

The question paper must not be removed from the examination room.

 TURN OVER

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ N m^2 kg^{-2}
Speed of light	c	$= 3 \times 10^8$ m s^{-1}
1 kpc		$= 3 \times 10^{19}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.
Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.
The metric signature $(+---)$ is used.
Partial derivatives are denoted by ",".
Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$u^i = dx^i/ds$ is the 4-velocity along the geodesic.

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \text{ where } R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl} \Gamma^n_{km} - \Gamma^i_{nm} \Gamma^n_{kl}.$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm}R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il}g^{km}R_{iklm} = g^{ik}R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4}T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right)c^2dt^2 - \frac{dr^2}{(1-\frac{r_g}{r})} - r^2\left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\text{ km}, \text{ where } M_\odot \text{ is the mass of Sun.}$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g rac}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Eikonal equation for photons:

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0,$$

where four-wave vector of the photon $k_i = -\frac{\partial\Psi}{\partial x^i}$.

Geodesic deviation equation:

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm}u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor.

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 Give the definition of a tensor with N contravariant and M covariant indices. What is the rank of the tensor and the number of independent components if $N = 2$ and $M = 3$. State the covariance principle and explain why according to this principle all physical equations should contains only tensors.

Question 2 Transformation from a local inertial (or local galilean) frame of reference $x^i_{(G)}$ to some non-inertial frame x^i is given by the following transformation matrix:

$$S^i_{(G)k} \equiv \frac{\partial x^i}{\partial x^i_{(G)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & A(x^m) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & B(x^m) \end{pmatrix},$$

where $A(x^m)$ and $B(x^m)$ are some functions of the coordinates x^m. Show that the metric in the non-inertial frame of reference x^i has the following form

$$ds^2 = ???.$$

[Hint: Express first g^{ik} in terms of the matrix $S^i_{(G)k}$ and the calculate g_{ik} taking into account that g^{ik} and g_{ik} are reciprocal with respect to each other.]

Question 3 Given that the interval

$$ds^2 = g_{ik} dx^i dx^k$$

is a scalar, prove that g_{ik} is a covariant tensor of the second rank. Show that without loss of generality this tensor has 10 independent components.

Question 4 Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero, $g^{ik}{}_{;n} = 0$.

Question 5 Using the Kerr metric given in the rubric, find the location of the event horizon, r_{hor}, and the limit of stationarity, r_{st}. Compare these results with the case of a non-rotating black hole. Give brief qualitative explain of what is the main difference between the limit of stationarity and the event horizon of a black hole.

Question 6 Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity,

$$\Omega_{hor} = \frac{a}{r_g r_{hor}}.$$

Question 7 The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that

$$g^{ik} p_i p_k = m^2 c^2.$$

Then, derive the Hamilton-Jacobi equation and explain how using this equation one can describe the motion of a test particle in a given gravitational field.

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8 (a) Using Biancci identity prove that covariant divergence of the Ricci tensor R_{ik} is related to the the gradient of the scalar curvature R by the following relationship:

$$R^i_{k\,;\,i} = \frac{1}{2} R_{,k}.$$

[12]

(b) Using the Einstein Field Equations (EFEs) given in the rubric and the identity proved in the previous sub-question show that the covariant divergence of the stress-energy tensor is equal to zero, $T^i_{k\,;\,k} = 0$. Explain briefly why this equation is considered as energy and momenta conservation law. [13]

(c) Take the stress-energy tensor in the form

$$T^i_k = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & -p & 0 & 0 \\ 0 & 0 & -p & 0 \\ 0 & 0 & 0 & -p \end{pmatrix},$$

where ε is energy density and p is pressure (if $p > 0$) or tension (if $p < 0$). Using the Einstein equations, evaluate the scalar curvature in terms of ε and p. [13]

Question 9 (a) Using the equation $ds = 0$ with $\theta,\ \phi = \text{const}$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r = r_0$ at time $t = 0$. Show that the world-line of the photon is given by

$$ct = r - r_0 + r_g \ln \frac{r - r_g}{r_0 - r_g}.$$

[5]

(b) A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then

$$r(\tau) = \left[r^{3/2}(\tau_0) - \frac{3}{2} c r_g^{1/2} (\tau - \tau_0) \right]^{2/3},$$

where τ is the proper time ($ds = c d\tau$).

[10]

(c) Using the coordinate transformations

$$c\tau = ct + \int \frac{r_g^{1/2} r^{1/2} dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2} dr}{r_g^{1/2}\,(r - r_g)}$$

show that the Schwarzschild metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r} dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Expressing r in terms of $R - c\tau$, demonstrate that the latter metric is non-stationary. What can be said about the true character of the Schwarzschild space-time metric at $r = r_g$?

[10]

Question 10 Consider the propagation of a photon in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzshild gravitational field.

(a) Derive the Eikonal equation

$$g^{ik} \frac{\partial \Psi}{\partial x^i} \frac{\partial \Psi}{\partial x^k} = 0$$

from the Hamiltom-Jacobi equation or otherwise. [10]

(b) Given that the solution of the Eikonal equation can be written in the following form

$$\Psi = -\omega t + \frac{\omega \rho}{c} \phi + \Psi_r(r),$$

where ω is the frequency of the photon and ϱ is its impact parameter, find a differential equation for Ψ_r and show that

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{dr}{cdt} = \sqrt{1 - \frac{\varrho^2}{r^2} + \frac{\varrho^2 r_g}{r^3}}.$$

[10]

(c) Sketch the regions of possible motions on the $(r - \rho)$ diagram and hence show that the radius of the unstable stable circular orbit for photons corresponds to $\varrho = \frac{3\sqrt{3}}{2} r_g$ and $r = \frac{3}{2} r_g$. [10]

Question 11 Consider a plane gravitational wave propagating along the x-axis. All components of $h_{ik} = g_{ik} - \eta_{ik}$ vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$.

Let two test particles be located in the $(y-z)$ plane and separated by the 3-vector $l^\alpha = (0, l_0 \cos\theta, l_0 \sin\theta)$.

(a) Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+ \cos 2\theta + h_\times \sin 2\theta).$$

[10]

(b) Consider a ring of test particles initially at rest in the $(y - z)$ plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}, \frac{\pi}{\omega}, \frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$. Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t-x/c)$. Finally consider the superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. What would you call this state of polarization? [10]

(c) Consider a binary system located in the center of our Galaxy ($R \approx 10kpc$), and consisting of two components of the same mass m. Show that to an order of magnitude the amplitude of the gravitational radiation generated by the binary and its frequency are $h_0 \sim r_g^2/(rR)$ and $\omega \sim (c r_g^{1/2} r^{-3/2})$ respectively, where r_g is the gravitational radius of each component and r is the separation between the two components. A future gravitational wave antenna detects gravitational radiation with frequency $10^{-3} Hz$ and amplitude 10^{-23}. Estimate the mass m and r. [10]

 END OF EXAMINATION

M. Sci. Examination by course unit 2009

MTH720U/MTHM033 Relativity and gravitation. SOLUTIONS

Duration: 3 hours

Date and time: xx xxx 2009, xxxxh

Apart from this page, you are not permitted to read the contents of this question paper until instructed to do so by an invigilator.

 TURN OVER

Section A: Each question carries 8 marks. You should attempt ALL questions.

Question 1 *Give the definition of a tensor with N contravariant and M covariant indices. What is the rank of the tensor and the number of independent components if $N = 2$ and $M = 3$. State the covariance principle and explain why according to this principle all physical equations should contains only tensors.*

Solution 1 *[Book work]*

This is the mixed tensor of $N + M$ rank defined as the object containing 4^{N+M} components $A^{i_1\, i_2\, i_3 \ldots i_N}_{j_1\, j_2\, j_3 \ldots j_M}$ which in the course of an arbitrary transformation from one frame of reference, x'^n, to another, x^m, are transformed according to the following transformation law:

$$A^{i_1\, i_2\, i_3 \ldots i_N}_{j_1\, j_2\, j_3 \ldots j_M} = \underbrace{S^{i_1}_{m_1} S^{i_2}_{m_2} S^{i_3}_{m_3} \ldots S^{i_N}_{m_N}}_{N \text{ times}} \underbrace{\tilde{S}^{n_1}_{j_1} \tilde{S}^{n_2}_{j_2} \tilde{S}^{n_3}_{j_3} \ldots \tilde{S}^{n_M}_{j_M}}_{M \text{ times}} A'^{m_1\, m_2\, m_3 \ldots m_N}_{n_1\, n_2\, n_3 \ldots n_M},$$

where

$$S^l_m = \frac{\partial x^l}{\partial x'^m} \text{ and } \tilde{S}^l_m = \frac{\partial x'^l}{\partial x^m}.$$

This tensor of the fifth rank and contains in the most general case $4^5 = 1024$ components. [5]

The Principle of Covariance says: The shape of all physical equations should be the same in an arbitrary frame of reference. Otherwise the physical equations [being different in gravitational field and in inertial frames of reference] would have different solutions. Laws of transformations for tensors and only for tensors keep the the shape of equations unchanged after transformations of coordinates. [3]

Question 2 *Transformation from a local inertial (or local Galilean) frame of reference $x^i_(G)$ to some non-inertial frame x^i is given by the following transformation matrix:*

$$S^i_{(G)k} \equiv \frac{\partial x^i}{\partial x^i_{(G)}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + A(x^m) & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $A(x^m) \neq -1$ is a function of the coordinates x^m. Show that the metric in the non-inertial frame of reference x^i has the following form

$$ds^2 = (dx^0)^2 - \frac{dx^1)^2}{(1 + A)^2} - (dx^2)^2 - (dx^3)^2.$$

.

[Hint: Express first g^{ik} in terms of the matrix $S^i_{(G)k}$ and then calculate g_{ik} taking into account that g^{ik} and g_{ik} are reciprocal with respect to each other.]

Solution 2 *[Unseen]*

Let us first calculate g^{ik}. One can rewrite $S^i_{(G)k}$ as

$$S^i_{(G)k} = \delta^i_k + A\delta^i_1\delta^1_k.$$

Taking into account that

$$g^{ik}_{(G)} = \eta^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

we have

$$g^{ik} = S^i_{(G)n} S^k_{(G)m} \eta^{nm} = (\delta^i_n + A\delta^i_1\delta^1_n)(\delta^k_m + A\delta^k_1\delta^1_m)\eta^{nm} =$$

$$= [\delta^i_n\delta^k_m + A(\delta^i_n\delta^k_1\delta^1_m + \delta^k_m\delta^i_1\delta^1_n) + A^2\delta^i_1\delta^1_n\delta^k_1\delta^1_m]\eta^{nm} =$$

$$= \eta^{in} + A(\eta^{i1}\delta^k_1 + \eta^{1k}\delta^i_1) + A^2\eta^{11}\delta^i_1\delta^k_1 = \eta^{in} - A(\delta^i_1\delta^k_1 + \delta^k_1\delta^i_1) - A^2\delta^i_1\delta^k_1 = \eta^{in} - 2A\delta^i_1\delta^k_1 - A^2\delta^i_1\delta^k_1 =$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+A)^2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

[5]

Determinant

$$|g^{ik}| = -(1+A^2) \neq 0,$$

hence g_{ik} which is reciprocal to g^{ik},is presented by inverse matrix:

$$g_{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -(1+A)^{-2} & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally

$$ds^2 = (dx^0)^2 - \frac{dx^1)^2}{(1+A)^2} - (dx^2)^2 - (dx^3)^2.$$

[3]

Question 3 *Given that the interval*

$$ds^2 = g_{ik}dx^i dx^k,$$

is a scalar. Prove that g_{ik} is a covariant tensor of the second rank. Show that without loss of generality this tensor has 10 independent components.

Solution 3. *[Seen similar]*

The fact that ds^2 is a scalar means

$$ds^2 = ds'^2, \;\; g_{ik}dx^i dx^k = g'_m n dx'^m dx'^n,$$

hence

$$0 = g_{ik}dx^i dx^k - g'_{mn}dx'^m dx'^n = g_{ik}dx^i dx^k - g'_{mn}\tilde{S}^m_i dx^i \tilde{S}^n_k dx^k = (g_{ik} - g'_{mn}\tilde{S}^m_i\tilde{S}^n_k)dx^i dx^k.$$

taking into account that dx^i and dx^k are arbitrary we conclude that the expression in brackets is equal to zero, hence

$$g_{ik} = g'_{mn}\tilde{S}^m_i\tilde{S}^n_k = \tilde{S}^m_i\tilde{S}^n_k g'_{mn},$$

thus according to the definition of the covariant second rank tensor g_{ik} indeed is the tensor of the second rank. [4]

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ik}dx^i dx^k}{2} = \frac{g_{ik}dx^i dx^k + g_{ik}dx^k dx^i}{2}.$$

The following substitution in the second term

$$i \to k \;\; k \to i$$

gives

$$ds^2 = \frac{g_{ik}dx^i dx^k + g_{ki}dx^i dx^k}{2} = \frac{g_{ik} + g_{ki}}{2} dx^i dx^k = \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{g_{ik} + g_{ki}}{2}.$$

Obviously that

$$\tilde{g}_{ik} = \tilde{g}_{ki}.$$

We can use $\tilde{g}_{ik}$ instead g_{ik} and then changing notations just drop $\tilde{}$. [4]

Question 4 *Using the formulae for the Cristoffel symbol and covariant derivatives given in the rubric or otherwise, show that covariant derivatives of contrvariant metric tensor are equal to zero,* $g^{ik}_{;n} = 0$.

Solution 4. *[Book work]*

The shortest proof (otherwise) looks like this. Let us first prove that $Dg^{ik} = 0$. Let A_i is an arbitrary covariant vector. By the definition of D one can say that DA_i is also vector and its contravariant representation is

$$DA^i = g^{ik} DA_k.$$

On other hand

$$DA^i = D(g^{ik}A_k) = Dg^{ik}A_k + g^{ik}DA_k,$$

hence

$$g^{ik}DA_k = Dg^{ik}A_k + g^{ik}DA_k$$

which means that

$$Dg^{ik}A_k = 0$$

for arbitrary vector A_k, hence

$$Dg^{ik} = 0.$$

[5]

By definition of covariant derivatives

$$Dg^{ik} = g^{ik}_{;m} dx^m$$

for arbitrary infinitesimally small displacement dx^m which means that

$$g^{ik}_{;m} = 0.$$

[3]

[The proof with the help of expressions for Γ^i_{kn} *is approximately twice longer.]*

Question 5 *Using the Kerr metric given in the rubric, find the location of the outer event horizon, r_{hor}, and the outer limit of stationarity, r_{st}. Give a brief qualitative explanation what is the main difference between the limit of stationarity and the event horizon of a black hole.*

Solution 5. *[Book work]*

Location of the event horizon corresponds to

$$g^{11} = 0.$$

Taking into account that all out of diagonal components $g_{1i} = 0$ (if $i \neq i$), one can see that

$$g^{11} = \frac{1}{g_{11}}$$

and the location of event horizon can be determined from

$$g_{11} = \infty$$

or, as follows from the expressions for Kerr metric given in the rubric, from

$$\Delta = r^2 - r_g r + a^2 = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2}}{2}.$$

The outer event horizon,r_{hor} corresponds to the sign "+", hence

$$r_{hor} = \frac{r_g}{2}\left(1 + \sqrt{1 - \alpha^2}\right),$$

where

$$\alpha = \frac{2a}{r_g} = \frac{ac^2}{GM}.$$

[4]

The location of limit of stationarity corresponds to

$$g_{00} = 0.$$

In the case of Kerr metric this corresponds to

$$1 - \frac{r_g r}{\rho^2} = 0,$$

where

$$\rho^2 = r^2 + a^2 \cos^2\theta,$$

hence from

$$r^2 - r_g r + a^2 \cos^2\theta = 0.$$

There are two solutions

$$r_\pm = \frac{r_g \pm \sqrt{r_g^2 + 4a^2 \cos^2\theta}}{2}.$$

The outer limit of stationarity, r_{st} corresponds to the sign "+", hence

$$r_{st} = \frac{r_g}{2}\left(1 + \sqrt{1 - \alpha^2 \cos^2\theta}\right).$$

[3]

Within the limit of stationarity no test particle can be in rest, but it does not mean that such a particle should move inward. Within the event horizon all particles should move inward. [1]

Question 6 *Show that the circle defined by $r = r_{hor}$ and $\theta = \pi/2$, is the world line of a photon moving around the rotating black hole with angular velocity*

$$\frac{d\phi}{dt} = \Omega_{hor} = \frac{ac}{r_g r_{hor}}.$$

Solution 6. *[Unseen]*

Putting $dr = d\theta = 0$, $r = r_{hor}$ and $\theta = \pi/2$ (i.e. $\sin\theta = 1$, $\cos\theta = 0$ and $\rho = r$) into the Kerr metric, [1]
one obtains

$$ds^2 = (1 - \frac{r_g r}{\rho^2})c^2dt^2 - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+\frac{2r_g rac}{\rho^2}\sin^2\theta d\phi dt = (1 - \frac{r_g}{r_{hor}})c^2dt^2 - (r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}})d\phi^2 + \frac{2r_g ac}{r_{hor}}d\phi dt =$$

[3]

$$= [(1 - \frac{r_g}{r_{hor}})c^2 - \Omega_{hor}^2(r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g ac}{r_{hor}}\Omega_{hor}]dt^2 =$$

$$= [1 - \frac{r_g}{r_{hor}} - (\frac{a}{r_g r_{hor}})^2(r_{hor}^2 + a^2 + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g a}{r_{hor}}\frac{a}{r_g r_{hor}}]c^2dt^2 =$$

$$= [1 - \frac{r_g}{r_{hor}} - (\frac{a}{r_g r_{hor}})^2(r_{hor}r_g + \frac{r_g a^2}{r_{hor}}) + \frac{2r_g a}{r_{hor}}\frac{a}{r_g r_{hor}}]c^2dt^2 = [1 - \frac{r_g}{r_{hor}} - (\frac{a^2}{r_g r_{hor}})(1 + \frac{a^2}{r_{hor}^2}) + \frac{2a^2}{r_{hor}^2}]c^2dt^2 =$$

$$= (1 - \frac{r_g}{r_{hor}} - \frac{a^2}{r_{hor}^2} + \frac{2a^2}{r_{hor}^2})c^2dt^2 = (1 - \frac{r_g}{r_{hor}} + \frac{a^2}{r_{hor}^2})c^2dt^2 = (r_{hor}^2 - r_g r_{hor} + a^2)\frac{c^2dt^2}{r_{hor}^2} = 0.$$

[3]

The fact that $ds^2 = 0$ means that this is the world line of a photon. [1]

Question 7 *The four-velocity and the four-momentum of a particle of mass m in a gravitational field are defined as*

$$u^i = \frac{dx^i}{ds}, \quad p^i = mcu^i.$$

Show that

$$g^{ik}p_i p_k = m^2c^2.$$

Then derive the Hamilton-Jacobi equation and explain how using this equation one can describe the motion of a test particle in given gravitational field.

Solution 7. *[Book work]*

From

$$ds^2 = g_i k dx^i dx^k$$

we have

$$g_{ik} u^i u^k = g_{ik} \frac{dx^i}{ds} \frac{dx^k}{ds} = \frac{g_{ik} dx^i dx^k}{ds^2} = \frac{ds^2}{ds^2} = 1.$$

After this we have

$$g^{ik} p_i p_k = p^k p_k = g_{kj} p^k p^j = m^2 c^2 g_{kj} u^k u^j = m^2 c^2.$$

[3]

For each covariant vector p_i we can find such a scalar that

$$p_i = -S_{,i}.$$

Substituting this to the previous formula we obtain the following equation for S:

$$g^{ik} S_{,i} S_{,k} = m^2 c^2.$$

This equation is called the Hamilton-Jacobi equation. [2]

The Hamilton-Jacobi equations "works" in the following way:

(i) We solve this single equation for single scalar field $S(x^m)$;

(ii) Taking partial derivatives we calculate covariant components of the four-momenta vector

$$p_i = -S_{,i};$$

(iii) With the help of g^{ik} we obtain contravariant components of the four-momenta vector

$$p^k = g^{ki} p_i;$$

(v) Then we calculate components of the four-velocity

$$u^i = \frac{p^i}{mc};$$

(vi) Finally we calculate world lines of test particles

$$x^i(s) = \int u^i ds.$$

[3]

Section B: Each question carries 22 marks. You may attempt all questions. Except for the award of a bare pass, only marks for the best TWO questions will be counted.

Question 8

(a) *Using Bianchi identity prove that covariant divergence of the Ricci tensor R_{ik} tensor is related to the the gradient of the scalar curvature R by the following relationship:*

$$R^i_{k\,;\,i} = \frac{1}{2}R_{,k}.$$

[14]

Solution 8.a *[book work]*

After contracting the Bianchi identity

$$R^i_{klm;n} + R^i_{knl;m} + R^i_{kmn;l} = 0$$

over indices i and n (taking summation $i = n$) we obtain

$$R^i_{klm;i} + R^i_{kil;m} + R^i_{kmi;l} = 0.$$

[1]

According to the definition of Ricci tensor

$$R^i_{kil} = R_{kl},$$

the second term can be rewritten as

$$R^i_{kil;m} = R_{kl;m}.$$

[1]

Taking into account that the Riemann tensor is antisymmetric with respect permutations of indices within the same pair

$$R^i_{kmi} = -R^i_{kim} = -R_{km},$$

the third term can be rewritten as

$$R^i_{kmi;l} = -R_{km;l}.$$

The first term can be rewritten as

$$R^i_{klm;i} = g^{ip}R_{pklm;i},$$

then taking mentioned above permutation twice we can rewrite the first term as

$$R^i_{klm;i} = g^{ip}R_{pklm;i} = -g^{ip}R_{kplm;i} = g^{ip}R_{kpml;i}.$$

After all these manipulations we have

$$g^{ip}R_{kpml;i} + R_{kl;m} - R_{km;l} = 0.$$

[2]

Then multiplying by g^{km} and taking into account that all covariant derivatives of the metric tensor are equal to zero, we have

$$g^{km}g^{ip}R_{kpml;i}+g^{km}R_{kl;m}-g^{km}R_{km;l} = \left(g^{km}g^{ip}R_{kpml}\right)_{;i}+\left(g^{km}R_{kl}\right)_{;m}-\left(g^{km}R_{km}\right)_{;l} = 0.$$

[3]

In the first term expression in brackets can be simplified as

$$g^{km}g^{ip}R_{kpml} = g^{ip}R_{pl} = R^i_l.$$

In the second term expression in brackets can be simplified as

$$g^{km}R_{kl} = R^m_l.$$

[2]

According to the definition of scalar curvature

$$R = g^{km}R_{km},$$

the third term can be simplified as

$$\left(g^{km}R_{km}\right)_{;l} = R_{;l} = R_{,l}.$$

[2]

Thus

$$R^i_{l;i} + R^m_{l;m} - R_{,l} = 0,$$

replacing in the second term index of summation m by i we finally obtain

$$2R^i_{l;i} - R_{,l} = 0, \text{ or } R^i_{l;i} - \frac{1}{2}R_{,l} = 0.$$

[3]

(b) *Using the EFEs given in the rubric and the identity proofed in the previous sub-question show that the covariant divergence of the stress-energy tensor is equal to zero,* $T^i_{k\,;\,k} = 0$. [4]

Solution 8.b*[seen similar]*

Multiplying the EFEs

$$R_{ik} - \frac{1}{2}g_{ik}R = \frac{8\pi G}{c^4}T_{ik}$$

by g^{mk} we obtain

$$R^m_i - \frac{1}{2}\delta^m_i R = \frac{8\pi G}{c^4}T^m_k.$$

[2]

Taking covariant divergence of LHS and RHS of this equation we obtain

$$R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m} = \frac{8\pi G}{c^4}T^m_{k;\,m},$$

hence

$$T^m_{k;\,m} = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}\delta^m_i R_{;m}\right) = \frac{c^4}{8\pi G}\left(R^m_{i;\,m} - \frac{1}{2}R_{,i}\right) = 0.$$

[2]

(c) *The stress-energy tensor has the following form*

$$T_{ik} = \begin{pmatrix} \varepsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix},$$

where ε is energy density and P is pressure (if $P > 0$) or tension (if $P < 0$). Using the Einstein equations express the scalar curvature in terms of ε and P. [4]

Solution 8.c *[Unseen]*

Contracting the EFEs written in mixed form (see the previous sub-question)we have

$$R^m_m - \frac{1}{2}\delta^m_m R = \frac{8\pi G}{c^4} T^m_m,$$

[2]

hence

$$R - \frac{4}{2}R = \frac{8\pi G}{c^4}T = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

hence

$$R - 2R = \frac{8\pi G}{c^4}(\varepsilon - 3P),$$

finally

$$R = -\frac{8\pi G}{c^4}(\varepsilon - 3P).$$

[2]

Question 9

(a) *Using the equation $ds = 0$ with $\theta,\ \phi = const$, consider the propagation of radial light signals in the Schwarzschild space-time. Consider a photon emitted outward from $r = r_0$ at time $t = 0$. Show that the world-line of the photon is given by*

$$ct = r - r_0 + r_g \ln\frac{r - r_g}{r_0 - r_g}.$$

[5]

Solution 9.a *[Unseen]*

From $ds = 0$ for $\theta,\ \phi = const$, we have

$$c^2(1 - \frac{r_g}{r})dt^2 - (1 - \frac{r_g}{r})^{-1}dr^2 = 0,$$

[2]

hence

$$cdt = (1 - \frac{r_g}{r})^{-1} dr = r(r - r_g)^{-1} dr = \int r(r - r_g)^{-1} dr =$$

$$= \int (r - r_g + r_g)(r - r_g)^{-1} dr = (r - r_g) + r_g \ln(r - r_g) + C.$$

If at $t = 0$ $r = r_0$, then

$$C = -[(r_0 - r_g) + r_g \ln(r_0 - r_g)],$$

and finally

$$ct = r - r_0 + r_g \ln \frac{r - r_g}{r_0 - r_g}.$$

[3]

(b) *A particle moves along a radial geodesic in the Schwarzschild metric. Using the expression for ds and an appropriate component of geodesic equation, show that if the particle starts to fall freely from infinity, then*

$$r(\tau) = \left[r^{3/2}(\tau_0) - \frac{3}{2} c r_g^{1/2} (\tau - \tau_0) \right]^{2/3},$$

where τ is the proper time ($ds = cd\tau$). [7]

Solution 9.b *[Seen similar]*

A particle moves along radial geodesic in the Schwarzschild metric, then

$$\frac{cd^2t}{ds^2} + \Gamma^0_{00} c^2 (\frac{dt}{ds})^2 + 2\Gamma^0_{01} c \frac{dt}{ds}\frac{dr}{ds} + \Gamma^0_{11} (\frac{dr}{ds})^2 = 0.$$

$$\Gamma^0_{00} = \frac{1}{2} g^{00} (g_{00,0} + g_{00,0} - g_{00,0}) = 0,$$

$$\Gamma^0_{01} = \frac{1}{2} g^{00} (g_{00,1} + g_{10,0} - g_{01,0}) = \frac{1}{2} g^{00} \frac{dg_{00}}{dr} = \frac{1}{2} (1 - \frac{r_g}{r})^{-1} \frac{d(1 - \frac{r_g}{r})}{dr} = \frac{r_g}{2r^2} (1 - \frac{r_g}{r})^{-1},$$

$$\Gamma^0_{11} = \frac{1}{2} g^{00} (g_{10,1} + g_{10,1} - g_{11,0}) = 0,$$

[2]

so we have

$$\frac{d^2t}{ds^2} + + \frac{r_g}{r^2} (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}\frac{dr}{ds} = 0,$$

or

$$\frac{dt}{ds}(\frac{dt}{ds}) + (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}\frac{d}{ds}(1 - \frac{r_g}{r}) = (1 - \frac{r_g}{r})^{-1} \frac{dt}{ds}[\frac{dt}{ds}(1 - \frac{r_g}{r})] = 0,$$

hence

$$\frac{dt}{ds}(1 - \frac{r_g}{r}) = C.$$

[2]

At infinity $\frac{dt}{ds} = c^{-1}$, hence $C = c^{-1}$. Substituting this into eq. for ds, we have

$$1 = (1 - \frac{r_g}{r})c^2(1 - \frac{r_g}{r})^{-2}c^{-2} - (1 - \frac{r_g}{r})^{-1}(\frac{dr}{ds})^2,$$

$$1 - \frac{r_g}{r} = 1 - (\frac{dr}{ds})^2 \Rightarrow (\frac{dr}{d\tau}) = -c\sqrt{\frac{r_g}{r}},$$

we take "$-$" for falling objects, then

$$\frac{2}{3}r^{3/2}(\tau) - r^{3/2}(\tau_0) = -cr_g^{1/2}(\tau - \tau_0),$$

and finally

$$r(\tau) = [r^{3/2}(\tau_0) - \frac{3}{2}cr_g^{1/2}(\tau - \tau_0)]^{2/3}.$$

[3]

(c) *Using the coordinate transformations*

$$c\tau = ct + \int \frac{r_g^{1/2}r^{1/2}dr}{r - r_g}, \quad R = ct + \int \frac{r^{3/2}dr}{r_g^{1/2}(r - r_g)}$$

show that the Schwarzschild metric takes the form

$$ds^2 = c^2 d\tau^2 - \frac{r_g}{r}dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

Expressing r in terms of $R - c\tau$, demonstrate that the latter metric is non-stationary. What can be said about the true character of the Schwarzschild space-time metric at $r = r_g$? [10]

Solution 9.c *[Book work]*

By differentiating

$$cd\tau = cdt + \frac{r_g^{1/2}r^{1/2}dr}{r - r_g}, \quad dR = cdt + \frac{r^{3/2}dr}{r_g^{1/2}(r - r_g)}.$$

Subtracting the first from the second we have

$$dR - cd\tau = \frac{dr}{r - r_g}\left(\frac{r^{3/2}}{r_g^{1/2}} - r_g^{1/2}r^{1/2}\right) =$$

$$= \frac{r^{1/2}dr}{(r - r_g)r_g^{1/2}}(r - r_g) = \left(\frac{r}{r_g}\right)^{1/2} dr,$$

hence

$$dr = \left(\frac{r_g}{r}\right)^{1/2}(dR - cd\tau).$$

Subtracting the first multiplied by r/r_g from the second we have

$$\frac{r}{r_g}cd\tau - DR = cdt(\frac{r}{r_g} - 1),$$

hence

$$cdt = \frac{crd\tau - r_g dR}{r - r_g}.$$

[3]

Then substituting the expressions for dr and cdt into ds^2 in the Schwarzschild form we obtain

$$ds^2 = \frac{r - r_g}{r}\left(\frac{rcd\tau - r_g dR}{r - r_g}\right)^2 - \frac{r_g}{r - r_g}(dR - cd\tau)^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \frac{1}{r - r_g}\left[\frac{1}{r}(rcd\tau - r_g dR)^2 - r_g(dR - cd\tau)^2\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= \left[c^2 d\tau^2 (r - r_g) - 2cdRd\tau(\frac{r_g r}{r} - r_g) - dR^2(\frac{r_g^2}{r} - r_g)\right] - r^2(d\theta^2 + \sin^2\theta d\phi^2) =$$

$$= c^2 d\tau^2 - \frac{r_g}{r}dR^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

From

$$r^{1/2}dr = r_g^{1/2}d(R - c\tau)$$

we have

$$\frac{2}{3}r^{3/2} = C + r_g^{1/2}(R - c\tau),$$

then choosing the constant of integration $C = 0$ so that $r = 0 \;\rightarrow\; R - c\tau = 0$, we have

$$r = \left[\frac{3}{2}r_g^{1/2}(R - c\tau)\right]^{2/3}.$$

[4]

Finally, putting this into the metric in new coordinates we have

$$ds^2 = c^2 d\tau^2 - \left[\frac{2r_g}{3(R - c\tau)}\right]^{2/3} - \left[\frac{3}{2}r_g^{1/2}(R - c\tau)\right]^{4/3}(d\theta^2 + \sin^2\theta d\phi^2),$$

we can see that the metric depends on τ, which means that the gravitational field is non- stationary. [2]

We can see that there is no physical singularity at $r = r_g$. [1]

Question 10 *Consider the propagation of a photon in the equatorial plane ($\theta = \frac{\pi}{2}$) of the spherically symmetric Schwarzschild gravitational field.*

(a) *Derive the Eikonal equation*

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0$$

from the Hamilton-Jacobi equation or otherwise. [4]

Solution 10.a *[Book work]*

The Eikonal equation can be obtained from Hamilton-Jacobi equation by setting $m = 0$ and replacing S by Ψ with $\frac{\partial \Psi}{\partial t} = -\omega$, where ω is frequency of the light, and replacing the constant angular momentum L by an impact parameter $\rho = cL/\omega$. [4]

[To derive this equation otherwise takes approximately the half of a page.]

(b) *Given that the solution of the Eikonal equation can be written in the following form*

$$\Psi = -\omega t + \frac{\omega\rho}{c}\phi + \Psi_r(r),$$

where ω is the frequency of the photon and ϱ is its impact parameter, find a differential equation for Ψ_r and show that

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{dr}{cdt} = \sqrt{1 - \frac{\varrho^2}{r^2} + \frac{\varrho^2 r_g}{r^3}}.$$

[12]

Solution 10.b *[Seen similar]*

Taking $\theta = \pi/2$ we can write down the Eikonal equation in the Schwarzschild metric as

$$\left(1 - \frac{r_g}{r}\right)^{-1} \left(\frac{\partial \Psi}{c\partial t}\right)^2 - \left(1 - \frac{r_g}{r}\right) \left(\frac{\partial \Psi}{\partial r}\right)^2 - \frac{1}{r^2} \left(\frac{\partial \Psi}{\partial \phi}\right)^2 = 0.$$

[3]

Then putting

$$\Psi = -\omega t + \frac{\omega\rho}{c}\phi + \Psi_r(r),$$

we have

$$\left(1 - \frac{r_g}{r}\right)^{-1} \frac{\omega^2}{c^2} - \left(1 - \frac{r_g}{r}\right) \left(\frac{d\Psi_r}{dr}\right)^2 - \frac{\rho^2\omega^2}{c^2 r^2} = 0,$$

which is the usual differential equation for $\Psi_r(r)$. [2]

The radial component of the four wave vector k^i can be found as (λ is a arbitrary scalar parameter along the world line of photon)

$$k^1 = \frac{dr}{d\lambda} = g^{11}k_1 = -g^{11}\frac{d\Psi}{dr} = (1 - \frac{r_g}{r})\sqrt{\frac{\omega^2}{c^2}(1 - \frac{r_g}{r}^{-2} - \frac{\rho^2\omega^2}{c^2r^2}(1 - \frac{r_g}{r})^{-1}} =$$

$$= \frac{\omega}{c}\sqrt{1 - \frac{\rho^2}{r^2}(1 - \frac{r_g}{r})} = \frac{\omega}{c}\sqrt{1 - \frac{\rho^2}{r^2} + \frac{\rho^2 r_g}{r^3}}.$$

[4]

On other hand

$$k^0 = \frac{cdt}{d\lambda} = g^{00}k_0 = -g^{00}\left(\frac{\partial \Psi}{c\partial t}\right) = g^{00}\frac{\omega}{c} = (1 - \frac{r_g}{r})^{-1}\frac{\omega}{c}.$$

Thus

$$\frac{dr}{cdt} = \frac{\frac{dr}{d\lambda}}{\frac{cdt}{d\lambda}} = \left(1 - \frac{r_g}{r}\right)\sqrt{1 - \frac{\rho^2}{r^2} + \frac{r_g \rho^2}{r^3}}.$$

[3]

(c) *Find the regions of possible motions on the $(r - \rho)$ diagram and show that the radius of the unstable stable circular orbit for photons corresponds to $\varrho = \frac{3\sqrt{3}}{2} r_g$ and $r = \frac{3}{2} r_g$.* [6]

Solution 10.c *[Seen similar]*

The limits of the radial motion (the turning points) are determined by the roots of the expression under the square root:

$$1 - \frac{\rho^2}{r^2} - \frac{r_g \rho^2}{r^3} = 0, \text{ hence } r^3 - \rho^2 r + \rho^2 r_g = 0,$$

thus

$$\rho^2 = \frac{r^3}{r - r_g} \text{ and } \rho = \pm \frac{r^{3/2}}{(r - r_g)^{1/2}}.$$

[2]

When $r \to \infty$, $\rho \to r$. When $r \to r_g$, $\rho \to \infty$. [1]

The curve $\rho(r)$ has a minimum corresponding to unstable circular orbit:

$$\frac{d\rho}{dr} = 0$$

gives

$$\frac{3r^2}{r - r_g} - \frac{r^3}{(r - r_g)^2} = 0 \text{ or } 3(r - r_g) - r = 0,$$

finally

$$r_* = \frac{3}{2} r_g$$

and

$$\rho_* = \left(\frac{3}{2} r_g\right)^{3/2} \frac{1}{(\frac{3}{2} - 1)^{1/2} r_g^{1/2}} = \frac{3\sqrt{3}}{2} r_g.$$

[3]

Question 11 *Consider a plane gravitational wave propagating along the x-axis. All components of $h_{ik} = g_{ik} - \eta_{ik}$ vanish except $h_{22} = -h_{33} \equiv h_+$ and $h_{23} = h_{32} = h_\times$. Let two test particles be located in the $(y - z)$ plane and separated by the 3-vector $l^\alpha = (0, l_0 \cos\theta, l_0 \sin\theta)$.*

(a) *Show that the perturbation of the distance δl between the two particles in the gravitational wave varies as*

$$\delta l = l - l_0 = \frac{l_0}{2}(h_+ \cos 2\theta + h_\times \sin 2\theta).$$

[4]

Solution 11.a*[Seen similar]*

$$l = \sqrt{-g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta} =$$

$$= \sqrt{-(\eta_{\alpha\beta} + h_{\alpha\beta})\Delta x^\alpha \Delta x^\beta} =$$

$$\sqrt{-\eta_{\alpha\beta}\Delta x^\alpha \Delta x^\beta - h_+(\Delta y^2 - \Delta z^2) - 2h_\times \Delta y \Delta x} =$$

$$= \sqrt{l_0^2 - h_+ l_0^2(\cos^2\theta - \sin^2\theta) - 2h_x l_0^2 \cos\theta \sin\theta} =$$

$$= l_0[1 - \frac{h_+}{2}\cos 2\Theta - \frac{h_\times}{2}\sin 2\Theta]$$

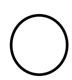
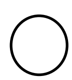
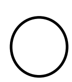
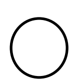
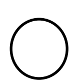
[3]

$$\delta l = l - l_0 = -\frac{1}{2} l_0 (h_+ \cos 2\theta + h_\times \sin 2\theta)$$

[1]

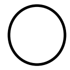
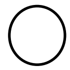
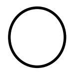
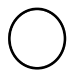
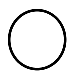
(b) *Consider a ring of test particles initially at rest in the $(y - z)$ plane and a plane monochromatic gravitational wave with frequency ω and polarization $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = 0$. Sketch the shape of the ring perturbed by the gravitational wave at times $t = \frac{\pi}{2\omega}$, $\frac{\pi}{\omega}$, $\frac{3\pi}{2\omega}$ and $\frac{2\pi}{\omega}$. Repeat the analysis for a gravitational wave with another polarization: $h_+ = 0$, $h_\times \sin\omega(t - x/c)$. Finally consider the superposition of two polarized waves: $h_+ = h_0 \sin\omega(t - x/c)$, $h_\times = h_0 \cos\omega(t - x/c)$. What would you call this state of polarization?* [10]

Solution 11.b*[Seen similar]*

ωt	$\delta l(\theta)$	$l(\theta)$
0	0	
$\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \cos 2\theta$	
π	0	
$\frac{3\pi}{2}$	$\frac{1}{2} l_0 h_0 \sin\omega t \cos 2\theta$	
2π	0	

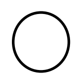
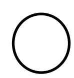
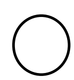
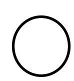
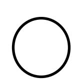
[3]

If $h_+ = 0$, $h_\times = h_0 \sin\omega t$ $\delta l(\theta) = -\frac{1}{2} l_0 h_0 \sin\omega t \sin 2\theta$

ωt	$\delta l(\theta)$	$l(\theta)$
0	0	
$\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \sin 2\theta$	
π	0	
$\frac{3\pi}{2}$	$\frac{1}{2} l_0 h_0 \sin 2\theta$	
2π	0	

[3]

If $h_+ = h_0 \sin\omega t$, $h_\times = h_0 \cos\omega t$
$\delta l(\theta) = -\frac{1}{2} l_0 h_0 (\sin\omega t \cos 2\theta + \cos\omega t \sin 2\theta) = -\frac{1}{2} l_0 h_0 (\sin\omega t + 2\theta) = -\frac{1}{2} l_0 h_0 \sin 2(\theta - \theta_0(t))$, where $\theta_0(t) = -\frac{1}{2}\omega t$

ωt	$\theta_0(t)$	$\delta l(\theta)$	$l(\theta)$
0	0	$-\frac{1}{2} l_0 h_0 \sin 2\theta$	
$\frac{\pi}{2}$	$-\frac{\pi}{4}$	$-\frac{1}{2} l_0 h_0 \sin 2(\theta + \frac{\pi}{4}) = -\frac{1}{2} l_0 h_0 \cos 2\theta$	
π	$-\frac{\pi}{2}$	$-\frac{1}{2} l_0 h_0 \sin 2(\theta + \frac{\pi}{2}) = \frac{1}{2} l_0 h_0 \sin 2\theta$	
$\frac{3\pi}{2}$	$-\frac{3\pi}{4}$	$\frac{1}{2} l_0 h_0 \cos 2\theta$	
2π	$-\pi$	$-\frac{1}{2} l_0 h_0 \sin(2\theta + 2\pi) = -\frac{1}{2} l_0 h_0 \sin 2\theta$	

[3]

This polarization can be called circular polarization. [1]

(c) *Consider a binary system located in the center of our Galaxy ($R \approx 10kpc$), and consisting of two components of the same mass m. Show that to an order of* *magnitude the amplitude of the gravitational radiation generated by the binary and its frequency are $h_0 \sim r_g^2/(rR)$ and $\omega \sim (c r_g^{1/2} r^{-3/2})$ respectively, where r_g is the gravitational radius of each component and r is the separation between the two components. A future gravitational wave antenna detects gravitational radiation with frequency $10^{-3}Hz$ and amplitude 10^{-23}. Estimate the mass m and r.* [8]

Solution 11.c *[Unseen]*

Using quadruple formula and taking into account that in the binary

$$x = r\cos\omega t, \quad y = r\sin\omega t,$$

where

$$\omega = (\frac{GM}{r^3})^{\frac{1}{2}},$$

we have

$$h \sim \frac{2G}{3c^4R}\omega^2 mr^2\cos 2\omega t \sim \frac{2G}{3c^4R}\frac{Gm}{r^3}mr^2\cos 2\omega t \sim \frac{r_g^2}{rR}\cos 2\omega t.$$

Hence

$$h \sim \frac{r_g^2}{rR}, \quad \omega \sim (\frac{GM}{r^3})^{\frac{1}{2}} \sim c\frac{r_g^{\frac{1}{2}}}{r^{\frac{3}{2}}}, \quad \frac{r_g^2}{r} \sim hR.$$

[3]

$$\frac{r_g^{\frac{1}{2}}}{r^{\frac{3}{2}}} \sim \frac{\omega}{c}, \quad r \sim \frac{r_g^2}{hR}, \quad \frac{r_g^{\frac{1}{2}}}{r^3}(hR)^{\frac{3}{2}} \sim \frac{\omega}{c}, \quad r_g^{-\frac{5}{2}} \sim \frac{\omega}{c}(hR)^{-\frac{3}{2}}.$$

$$r_g = (\frac{\omega}{c})^{-\frac{2}{5}}(hR)^{\frac{3}{5}} = (\frac{\omega R}{c})^{-\frac{2}{5}}Rh^{\frac{3}{5}} \sim (\frac{10^{-3}\cdot 10^4\cdot 3\cdot 10^18}{5\cdot 3\cdot 10^10})^{-\frac{2}{5}}\cdot 3\cdot 10^4\cdot 10^18\cdot 10^{-\frac{22\cdot 3}{5}} =$$

$$= 10^{-\frac{18}{5}}\cdot 3\cdot 10^22\cdot 10^{-\frac{66}{5}} \approx 3\cdot 10^{17-\frac{84}{5}} \approx 3\cdot 10^{\frac{1}{5}} \approx 3\cdot 1.4.$$

[3]

Hence $M = M_\odot(\frac{r_g}{3km}) \simeq 1.4M_\odot$ and

$$\frac{r}{r_g} \approx (\frac{3km\cdot 10^{\frac{1}{5}}}{10^{-22}\cdot 3\cdot 10^4\cdot 10^{13}km}) \approx 10^5\cdot 10^{\frac{1}{5}}.$$

$$r \simeq 5\cdot 10^5 km$$

[2]

End of Paper

Queen Mary
University of London

M.Sc. EXAMINATION

MAS 412 Relativity and Gravitation

29 April 2008 10:00-13:00
Duration: 3 hours

This paper has two Sections and you should attempt both Sections. Please read carefully the instructions given at the beginning of each Section.

Calculators ARE permitted in this examination. The unauthorized use of material stored in a pre-programmable memory constitutes an examination offence. Please state on your answer book the name and type of machine used.

Don't turn over the page until instructed to do so by an invigilator.

You are reminded of the following:

PHYSICAL CONSTANTS

Gravitational constant	G	$= 6.7 \times 10^{-11}$ N m^2 kg^{-2}
Speed of light	c	$= 3 \times 10^8$ m s^{-1}
1 kpc		$= 3 \times 10^{19}$ m

NOTATION

Three-dimensional tensor indices are denoted by Greek letters $\alpha, \beta, \gamma, ...$ and take on the values 1, 2, 3.

Four-dimensional tensor indices are denoted by Latin letters $i, k, l, ...$ and take on the values 0, 1, 2, 3.

The metric signature $(+ - --)$ is used.

Partial derivatives are denoted by ",".

Covariant derivatives are denoted by ";".

USEFUL FORMULAS, which you may use without proof.

Minkowski metric:

$$ds^2 = \eta_{ik} dx^i dx^k = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

Covariant derivatives:

$$A^i_{;k} = A^i_{,k} + \Gamma^i_{km} A^m, \quad A_{i;k} = A_{i,k} - \Gamma^m_{ik} A_m, \text{ where } \Gamma^i_{kn} \text{ are Christoffel symbols}$$

Christoffel symbols:

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(g_{mk,l} + g_{ml,k} - g_{kl,m} \right)$$

Geodesic equation:

$$\frac{du^i}{ds} + \Gamma^i_{kn} u^k u^n = 0,$$

where

$$u^i = dx^i/ds \text{ is the 4-velocity along the geodesic.}$$

Riemann tensor:

$$A^i_{;k;l} - A^i_{;l;k} = -A^m R^i_{mkl}, \text{ where } R^i_{klm} = g^{in} R_{nklm},$$

$$R^i_{klm} = \Gamma^i_{km,l} - \Gamma^i_{kl,m} + \Gamma^i_{nl}\Gamma^n_{km} - \Gamma^i_{nm}\Gamma^n_{kl}.$$

Symmetry properties of the Riemann tensor:

$$R_{iklm} = -R_{kilm} = -R_{ikml}, \quad R_{iklm} = R_{lmik}.$$

Bianchi identity:

$$R^n_{ikl;m} + R^n_{imk;l} + R^n_{ilm;k} = 0.$$

Ricci tensor:

$$R_{ik} = g^{lm} R_{limk} = R^m_{imk}.$$

Scalar curvature:

$$R = g^{il} g^{km} R_{iklm} = g^{ik} R_{ik} = R^i_i.$$

Einstein equations:

$$R^i_k - \frac{1}{2}\delta^i_k R = \frac{8\pi G}{c^4} T^i_k,$$

where T^i_k is the Stress-Energy tensor.

Schwarzschild metric:

$$ds^2 = \left(1 - \frac{r_g}{r}\right) c^2 dt^2 - \frac{dr^2}{\left(1 - \frac{r_g}{r}\right)} - r^2 \left(\sin^2\theta d\phi^2 + d\theta^2\right).$$

Gravitational radius:

$$r_g = 2GM/c^2 = 3(M/M_\odot)\ \text{km}, \quad \text{where}\ \ M_\odot\ \ \text{is the mass of Sun.}$$

Kerr metric:

$$ds^2 = (1 - \frac{r_g r}{\rho^2}) c^2 dt^2 - \frac{\rho^2}{\Delta} dr^2 - \rho^2 d\theta^2 - (r^2 + a^2 + \frac{r_g r a^2}{\rho^2}\sin^2\theta)\sin^2\theta d\phi^2$$

$$+ \frac{2 r_g r a c}{\rho^2}\sin^2\theta d\phi dt,$$

where $\rho^2 = r^2 + a^2\cos^2\theta$, $\Delta = r^2 - r_g r + a^2$, and $a = \frac{J}{mc}$, where J is the specific angular momentum.

Eikonal equation for photons:

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k} = 0,$$

where four-wave vector of the photon $k_i = -\frac{\partial\Psi}{\partial x^i}$.

Geodesic deviation equation:

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

Quadrupole formula for gravitational waves:

$$h_{\alpha\beta} = -\frac{2G}{3c^4 R}\frac{d^2 D_{\alpha\beta}}{dt^2},$$

where R is the distance to source of gravitational radiation and

$$D_{\alpha\beta} = \int (3x_\alpha x_\beta - r^2\delta_{\alpha\beta})dM$$

is the quadrupole tensor.

SECTION A

Each question carries 8 marks. You should attempt all questions.

1. State the equivalence principle and explain the difference between interpretation of this principle in Newtonian theory and in General relativity. State the covariance principle and explain the relationship between this principle and the principle of equivalence.

2. What is the reciprocal tensor? Demonstrate how, using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik}, you can form a contravariant tensor from covariant tensors and vice versa. Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

3. Give a rigorous proof that the interval squared,

$$ds^2 = g_{ik}dx^i dx^k,$$

is a scalar if it is given that g_{ik}, the metric tensor, is a covariant tensor of the second rank. Prove that the metric tensor is symmetric.

4. A light signal emitted at the moment corresponding to time coordinate $x^0 + \Delta x^{0(1)}$ propagates from some point B with spatial coordinates $x^\alpha + \Delta x^\alpha$ to a point A with spatial coordinates x^α and then, after reflection at the moment corresponding to time coordinate x^0, the signal propagates back over the same path and is detected at point B at the moment corresponding to time coordinate $x^0 + \Delta x^{0(2)}$. Given that $g_{0\alpha} = 0$, express the physical distance between A and B, l_{AB}, in terms of the metric tensor, g_{ik}, and Δx^α. You may assume that g_{ik} is the same at points A and B.

5. Show that all covariant derivatives of metric tensor are equal to zero. Find the relationship between the Christoffel symbols and first partial derivative of the metric tensor.

6. Explain the main difference between the limit of stationarity and the event horizon of a black hole?

7. Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? Compare your results with the case of the Schwarzschild black hole.

SECTION B

Each question carries 22 marks. Only marks for the best TWO questions will be counted.

1. (a) [**10 Marks**] Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

(b) [**8 Marks**] Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where G is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

(c) [**4 Marks**] What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

2. The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around a Schwarzschild black hole.

(a) [**5 Marks**] What is the physical meaning of the "effective potential energy"? Explain how U can be used to find stable and unstable circular orbits.

(b) [**10 Marks**] Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows:

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2r - 3r_g)^{1/2}\, r^{1/2}}.$$

(c) [**7 Marks**] Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

3. Consider a compact object of mass m moving along circular orbit around a black hole of mass M assuming that $m \ll M$ and using the quadrupole formula for the metric perturbation associated with gravitational waves,

(a) [**7 Marks**] show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such a system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period;

(b) [**9 Marks**] show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

(c) [**6 Marks**] The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4} Hz < \omega < 3 \cdot 10^{-3} Hz$. From what distance will it be possible to detect gravitational radiation from a binary system, containing a black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$, around a massive black hole of mass $M = 10^3 M_\odot$?

4. (a) **[8 Marks]** Derive the geodesic deviation equation

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics and u^k is the 4-velocity along the geodesic.

(b) **[9 Marks]** Consider two neighbouring particles freely falling from rest in the Schwarzschild gravitational field in the same radial direction. Using the geodesic deviation equation show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

(c) **[5 Marks]** If the height of an observer is $l \approx 2\text{m}$, find the radial distance $r \gg r_g$ from a solar mass neutron star at which the radial tidal 3-acceleration experienced by the observer at rest ($a = c^2 \frac{D^2\eta^1}{ds^2}$) is equal to 100 $g \approx 10^3\text{ms}^{-2}$. You may assume that the observer's body is aligned along the radial direction, and you may take the gravitational radius of the Sun to be 3 km.

 [End of examination paper.]

M.Sc. EXAMINATION

MAS 412 (MTHM 033) Relativity and Gravitation

xxx, xx May 2008 xx:xx-xx:xx
Duration: 3 hours

SOLUTIONS

SECTION A

1. Formulate the equivalence principle and explain what is the difference in interpretation of this principle in Newtonian theory and in General relativity. Formulate the covariance principle and explain the relationship between this principle and the principle of equivalence.

SOLUTION A1 [book work]

This principle states that an uniform gravitational field is equivalent to a uniform acceleration of reference frame.

[1/8 Mark]

In Newton theory the motion of a test particle is determined by the following equation of motion

$$m_{in}\vec{a} = -m_{gr}\nabla\phi,$$

where $\vec{a}$ is the acceleration of the test particle, ϕ is newtonian potential of gravitational field, m_{in} is the inertial mass of the test particle and m_{gr} is its gravitational mass. The fact that all test particles move with the same acceleration for given ϕ is explained within frame of newtonian theory just by the following "coincidence":

$$\frac{m_{in}}{m_g} = 1,$$

i.e. inertial mass m_{in} is equal to gravitational mass m_{gr}.

[2/8 Marks]

The General Relativity gives very simple and natural explanation of the Principle of Equivalence: In curved space-time all bodies move along geodesics, that is why their world lines are the same in given gravitational field. The situation is the same as in flat space-time when free particles move along straight lines which are geodesics in flat space-time.

[2/8 Marks]

The covariance principle says: The shape of all physical equations should be the same in an arbitrary frame of reference, including the most general case of non-inertial frames. If in contrast to the covariance principle the shape of physical equations were different in local inertial frames in presence of gravitational field and in non-inertial frames in absence of gravitational field then these equations would give different solutions, i.e. different predictions for (a) standing on the Earth, feeling the effects of gravity as a downward pull and (b) standing in a very smooth elevator that is accelerating upwards with the acceleration g , hence these equations would contradict to the basic postulate of the General Relativity, the principle of equivalence, which states that a uniform gravitational field (like that near the Earth) is equivalent to a uniform acceleration. Hence, the covariance principle is the mathematical formulation of the principle of equivalence.

[3/8 Marks]

2. Explain what is the reciprocal tensor. Demonstrate how using the reciprocal contravariant metric tensor g^{ik} and the covariant metric tensor g_{ik} you can form contravariant tensor from covariant tensors and vice versa. Show that in an arbitrary non-inertial frame

$$g^{ik} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3},$$

where $S^i_{(0)k}$ is the transformation matrix from locally inertial frame of reference (galilean frame) to this non-inertial frame.

SOLUTION A2 [book work]

Two tensors A_{ik} and B^{ik} are called reciprocal to each other if

$$A_{ik}B^{kl} = \delta^l_i.$$

[2/8 Marks]

We can introduce a contravariant metric tensor g^{ik} which is reciprocal to the covariant metric tensor g_{ik}:

$$g_{ik}g^{kl} = \delta^l_i.$$

With the help of the metric tensor and its reciprocal we can form contravariant tensor from covariant tensors and vice versa, for example:

$$A^i = g^{ik}A_k, \quad A_i = g_{ik}A^k.$$

[3/8 Marks]

We know that in the galilean frame of reference

$$g^{ik} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta^{ik} \equiv \mathrm{diag}(1, -1, -1, -1),$$

hence

$$g^{ik} = S^i_{(0)n}S^k_{(0)m}\eta^{lm} = S^i_{(0)0}S^k_{(0)0} - S^i_{(0)1}S^k_{(0)1} - S^i_{(0)2}S^k_{(0)2} - S^i_{(0)3}S^k_{(0)3}.$$

[3/8 Marks]

3. Give a rigorous proof that the interval squared,

$$ds^2 = g_{ik}dx^i dx^k,$$

is a scalar if given that g_{ik}, the metric tensor, is a covariant tensor of the second rank. Prove that the metric tensor is symmetric.

SOLUTION A3 [seen similar]

$$ds^2 = g_{ik} dx^i dx^k,$$

hence,

$$ds^2 = g_{ik} dx^i dx^k = (\tilde{S}_i^n \tilde{S}_k^m g'_{nm})(S_p^i dx'^p)(S_w^k dx'^w) = (\tilde{S}_i^n S_p^i)(\tilde{S}_k^m S_w^k)(g'_{nm} dx'^p dx'^w) =$$

$$= \delta_p^n \delta_w^m (g'_{nm} dx'^p dx'^w) = g'_{pw} dx'^p dx'^w = g'_{ik} dx'^i dx'^k = ds'^2,$$

thus

$$ds = ds'$$

which means that ds is a scalar.

[5/8 Marks]

$$ds^2 = g_{ik} dx^i dx^k = \frac{1}{2}(g_{ik} dx^i dx^k + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} dx^k dx^i + g_{ik} dx^i dx^k) = \frac{1}{2}(g_{ki} + g_{ik}) dx^i dx^k =$$

$$= \tilde{g}_{ik} dx^i dx^k,$$

where

$$\tilde{g}_{ik} = \frac{1}{2}(g_{ki} + g_{ik}),$$

which is obviously symmetric one. Then we just drop "~".

[3/8 Marks]

4. A light signal emitted at the moment corresponding to time coordinate $x^0 + \Delta x^{0(1)}$ propagates from some point B with spatial coordinates $x^\alpha + \Delta x^\alpha$ to a point A with spatial coordinates x^α and then after reflection at the moment corresponding to time coordinate x^0 the signal propagates back over the same path and is detected in the point B at the moment corresponding to time coordinate $x^0 + \Delta x^{0(2)}$. Given that $g_{0\alpha} = 0$, express the physical distance between A and B, l_{AB}, in terms of the metric tensor, g_{ik}, and Δx^α. You may assume that g_{ik} is the same in the points A and B.

SOLUTION A4 [seen similar]

For the proper time between any two events occurring at the same point in space we have

$$\tau = \frac{1}{c}\int \sqrt{g_{00}}dx^0.$$

[2/8 Marks]

Separating the space and time coordinates in ds we have

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta + 2g_{0\alpha}dx^0 dx^\alpha + g_{00}(dx^0)^2 = g_{\alpha\beta}dx^\alpha dx^\beta + g_{00}(dx^0)^2.$$

The interval between the events which belong to the same world line of light in Special and General Relativity is always equal to zero:

$$ds = 0.$$

Solving this equation with respect to dx^0 we find two roots:

$$dx^{0(1)} = -\frac{1}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}$$

and

$$dx^{0(2)} = \frac{1}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta},$$

hence

$$dx^{0(2)} - dx^{0(1)} = \frac{2}{g_{00}}\sqrt{-g_{\alpha\beta}g_{00})dx^\alpha dx^\beta}.$$

Then

$$dl = \frac{c}{2}d\tau = \frac{c}{2}\frac{\sqrt{g_{00}}}{c}(dx^{0(2)} - dx^{0(1)})$$

and finally

$$dl^2 = -g_{\alpha\beta}dx^\alpha dx^\beta,$$

and finally

$$l_{AB} = \int_B^A \sqrt{dl} = \sqrt{-g_{\alpha\beta}\Delta x^\alpha \Delta x^\beta}.$$

[6/8 Marks]

5. Show that all covariant derivatives of metric tensor are equal to zero. Find the relationship between the Cristoffel symbols and first partial derivative of the metric tensor.

SOLUTION A5 [book work]

$$DA_i = g_{ik}DA^k$$
$$DA_i = D(g_{ik}A^k) = g_{ik}DA^k + A^kDg_{ik},$$

hence

$$g_{ik}DA^k = g_{ik}DA^k + A^kDg_{ik},$$

which obviously means that

$$A^kDg_{ik} = 0.$$

Taking into account that A^k is arbitrary vector, we conclude that

$$Dg_{ik} = 0.$$

Then taking into account that

$$Dg_{ik} = g_{ik;m}dx^m = 0$$

for arbitrary infinitesimally small vector dx^m we have

$$g_{ik;m} = 0.$$

[3/8 Marks]

Introducing useful notation

$$\Gamma_{k,\, il} = g_{km}\Gamma^m_{il},$$

we have

$$g_{ik;\, l} = \frac{\partial g_{ik}}{\partial x^l} - g_{mk}\Gamma^m_{il} - g_{im}\Gamma^m_{kl} = \frac{\partial g_{ik}}{\partial x^l} - \Gamma_{k,\, il} - \Gamma_{i,\, kl} = 0.$$

Permuting the indices i, k and l twice as

$$i \to k, \quad k \to l, \quad l \to i,$$

we have

$$\frac{\partial g_{ik}}{\partial x^l} = \Gamma_{k,\, il} + \Gamma_{i,\, kl}, \quad \frac{\partial g_{li}}{\partial x^k} = \Gamma_{i,\, kl} + \Gamma_{l,\, ik} \;\text{ and }\; -\frac{\partial g_{kl}}{\partial x^i} = -\Gamma_{l,\, ki} - \Gamma_{k,\, li}.$$

Taking into account that

$$\Gamma_{k,\, il} = \Gamma_{k,\, li},$$

after summation of these three equation we have

$$g_{ik,l} + g_{li,k} - g_{kl,i} = 2\Gamma_{i,\,kl},$$

and finally

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right).$$

[5/8 Marks]

6. Explain what is the main difference between the limit of stationarity and the event horizon of a black hole?

SOLUTION A6 [book work]

The Limit of stationarity (Static Limit): the interval ds for test particle in rest

$$dr = d\theta = d\phi = 0.$$

In this case

$$ds^2 = g_{00} dx^{0^2},$$

We can see that if

$$g_{00} = 0,$$

then

$$ds^2 = 0,$$

which means that the world line of particle in rest is the world line of light. Hence, at the surface

$$g_{00} = 0$$

no particle with finite rest mass can be in rest. For this reason this surface is called the limit of stationarity.

[3/8 Marks]

Event Horizon is a spherically symmetric surface

$$F(r) = \text{const.}$$

Its normal vector is defined as usually as

$$n_i = F_{,i} = \delta^1_i \frac{dF}{dr}.$$

If at this surface

$$g^{11} = 0$$

then

$$g^{ik} n_i n_k = g^{11} n_1 n_1 = g^{11} \left(\frac{dF}{dr} \right)^2 = 0,$$

which means that n_i is a null vector and any particle with finite rest mass can not move outward the surface $g^{11} = 0$, thus this surface is the event horizon.

[5/8 Marks]

7. Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? (compare your results with the case of the Schwarzschild black hole).

SOLUTION A7 [seen similar]

Consider a rotating black hole described by the Kerr metric. Find the locations of event horizon, "limit of stationarity" and the "ergosphere"? (compare your results with the case of the Schwarzschild black hole). Describe briefly the Penrose process of extraction of energy from a rotating black hole and explain why this mechanism does not contradict to the statement, that nothing can escape from within black hole.

For the Kerr metric $g_{00} = 0$ gives

$$1 - \frac{r_g r}{\rho^2} = 0,$$

thus

$$r^2 - r_g r + a^2 \cos^2\theta = 0,$$

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r_{st} = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

[2/8 Marks]

The location of horizon in the Kerr metric: $g^{11} = 0$ ($g_{11} = \infty$) corresponds to

$$\Delta = r^2 - r_g r + a^2 = 0,$$

and

$$r = \frac{1}{2}(r_g \pm \sqrt{r_g^2 - 4a^2\cos^2\theta}) = \frac{r_g}{2} \pm \sqrt{(\frac{r_g}{2})^2 - a^2\cos^2\theta}.$$

$$r_{hor} = \frac{r_g}{2} \pm \sqrt{\left(\frac{r_g}{2}\right)^2 - a^2}.$$

[2/8 Marks]

One can see easily that

$$r_{st} \geq r_{hor},$$

for example,

$$r_{st} = r_{hor}, \text{ if } \theta = 0, \text{ or } \theta = \pi \text{ (at the poles)},$$

and

$$r_{st} = 2r_g > r_{hor}, \text{ if } \theta = \frac{\pi}{2} \text{ (at the equator).}$$

The region between the limit of stationarity and the event horizon is called the "ergosphere".

[3/8 Marks]

In the Schwarzschild metric as one can see putting $a = 0$,

$$r_{hor} = r_{st},$$

which means that in this case the "ergosphere" does not exist.

[1/8 Marks]

SECTION B

Each question carries 22 marks. Only marks for the best TWO questions will be counted.

1. (a) **[10 Marks]** Give the definition of the Ricci tensor R_{ik} and prove that

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}$$

SOLUTION B1(a) [Seen similar]

By definition the Ricci tensor is

$$R_{ik} = g^{lm} R_{limk} = R^l_{ilk},$$

where the curvature Riemann tensor is defined by

$$A_{i;\,k;\,l} - A_{i;\,l;\,k} = A_m R^m_{ikl}.$$

By straightforward calculations

$$A_{i;k;l} - A_{i;l;k} =$$

$$A_{i;k,l} - \Gamma^m_{li} A_{m;k} - \Gamma^m_{lk} A_{i;m} -$$

$$-A_{i;l,k} + \Gamma^m_{ki} A_{m;l} + \Gamma^m_{kl} A_{i;m} =$$

$$\left(A_{i,k} - \Gamma^m_{ik} A_m\right)_{,l} - \Gamma^m_{li}\left(A_{m,k} - \Gamma^n_{mk} A_n\right) -$$

$$-\left(A_{i,l} - \Gamma^m_{il} A_m\right)_{,k} + \Gamma^m_{ki}\left(A_{m,l} - \Gamma^n_{ml} A_n\right) =$$

$$A_{i,k,l} - A_{i,l,k} - \Gamma^m_{ik} A_{m,l} - \Gamma^m_{il} A_{m,k} - \Gamma^m_{kl} A_{i,m} + \Gamma^m_{il} A_{m,k} + \Gamma^m_{ik} A_{m,l} + \Gamma^m_{lk} A_{i,m} -$$

$$-\Gamma^m_{ik,l} A_m + \Gamma^m_{il}\Gamma^p_{mk} A_p + \Gamma^m_{kl}\Gamma^p_{im} A_p +$$

$$+\Gamma^m_{ik,l} A_m - \Gamma^m_{ik}\Gamma^p_{ml} A_p - \Gamma^m_{lk}\Gamma^p_{im} A_p =$$

$$= A_m \left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^p_{kl}\Gamma^m_{ip} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl} - \Gamma^p_{lk}\Gamma^m_{ip}\right) =$$

$$= A_m \left(-\Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} + \Gamma^m_{il,k} - \Gamma^p_{ik}\Gamma^m_{pl}\right),$$

hence

$$R^m_{ikl} = \Gamma^m_{il,k} - \Gamma^m_{ik,l} + \Gamma^p_{il}\Gamma^m_{pk} - \Gamma^p_{ik}\Gamma^m_{pl},$$

and replacing k by l and l by k and then just putting $m = l$ we finally obtain

$$R_{ik} = \frac{\partial \Gamma^l_{ik}}{\partial x^l} - \frac{\partial \Gamma^l_{il}}{\partial x^k} + \Gamma^l_{ik}\Gamma^m_{lm} - \Gamma^m_{il}\Gamma^l_{km}.$$

 [This question continues overleaf ...]

(b) [**8 Marks**] Starting from the Einstein equations in the form

$$R_{ik} - \frac{1}{2} g_{ik} R = \frac{8\pi G}{c^4} T_{ik},$$

where G is the gravitational constant, prove that

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

SOLUTION B1(b) [seen similar]

Contracting with g^{ik}, we have the Einstein equations in mixed form

$$R^i_k = \frac{8\pi G}{c^4}(T^i_k - \frac{1}{2}\delta^i_k T).$$

$$R = g^{ik} R_{ik} = \frac{8\pi G}{c^4}\left(g^{ik} T_{ik} - \frac{1}{2} g^{ik} g_{ik} T\right) = \frac{8\pi G}{c^4}\left(T^i_i - \frac{1}{2}\delta^i_i T\right) = \frac{8\pi G}{c^4}\left(T - \frac{1}{2}4\right) = -\frac{8\pi G}{c^4} T.$$

Thus

$$T = -\frac{c^4}{8\pi G} R.$$

Thus

$$T_{ik} = \frac{c^4}{8\pi G}\left(R_{ik} - \frac{1}{2} g_{ik} R\right),$$

then in mixed form we have

$$T^i_k = \frac{c^4}{8\pi G}\left(R^i_k - \frac{1}{2}\delta^i_k R\right).$$

(c) [**4 Marks**] c) What can you say about the nature of gravitational field, for which $R_{ik} = 0$, while R_{ikln} is not equal to zero?

SOLUTION B1(c) [unseen]

This situation corresponds to gravitational fields (for example, gravitational waves), when the space-time is curved, but matter is absent (empty space-time).

2. The "effective potential energy" is defined as

$$U(r) = mc^2 \left(1 - \frac{r_g}{r}\right)^{1/2} \left(1 + \frac{L^2}{m^2c^2r^2}\right)^{1/2},$$

where L is the angular momentum and m is the mass of a particle, moving around Schwarzschild black hole.

(a) [**5 Marks**] What is the physical meaning of the "effective potential energy"? Explain how using U to find stable and unstable circular orbits.

SOLUTION B2(a)[book work]

The effective potential energy includes potential energy and that part of kinetic energy, which is related with non-radial, angular motion. Points at which $E = U$, (E is the conservative total energy) correspond to turning points, where $dr/dt = 0$.

$$U = E, \;\; U_r^{'} = 0,$$

corresponds to the circular orbit, stable, if $U_{rr}^{''} > 0$, and unstable, if $U_{rr}^{''} < 0$.

(b) [**10 Marks**] Using the Hamilton-Jacobi equation, show that the energy of a particle moving along circular orbit depends on the radius of the orbit as follows

$$E(r) = \sqrt{2}mc^2 \frac{(r - r_g)}{(2r - 3r_g)^{1/2}\, r^{1/2}}.$$

SOLUTION B2(b)[seen similar]

Introducing $x = r_g/r$, we have $U_r^{'} = 0$ corresponds $U_x^{'} = 0$, so

$$[(1-x)(1+\alpha x^2]_x^{'} = 0,$$

where

$$\alpha = \frac{L^2}{m^2c^2r_g^2},$$

$$-1 - 3\alpha x^2 + 2\alpha x = 0,$$

and

$$\alpha = \frac{1}{x(2-3x)}.$$

Then

$$\frac{E^2}{m^2c^4} = (1-x)(1 + \frac{x}{2-3x}) = \frac{2(1-x)^2}{3-3x},$$

and finally

$$E = \frac{\sqrt{2}mc^2(1 - r_g/r)}{(2 - 3r_g/r)^{1/2}} = \frac{\sqrt{2}mc^2(r - r_g)}{(2r - 3r_g)^{1/2}r^{1/2}}.$$

(c) **[7 Marks]** Determine the radius of the last circular orbit. What fraction of the initial energy will be released by the particle when it reaches the last circular orbit?

SOLUTION B2(c)[unseen]

The last circular orbit corresponds the following system of equations: $E = U$, $U' = 0$, $U'' = 0$.

$$0 = U'' \sim 2\alpha(1 - 3x),$$

so $x = 1/3$, which corresponds to $r = 3r_g$.

$$\frac{E^2}{m^2c^4} = (1 - 1/3)(1 + 3/3^2) = 8/9,$$

and

$$E_{lo} = mc^2 \frac{2\sqrt{2}}{3}.$$

Fraction of energy:

$$f = \frac{E_\infty - E_{lo}}{E_\infty} = 1 - \frac{2\sqrt{2}}{3} = 0.057$$

3. Consider a compact object of mass m moving along circular orbit around the black hole of mass M, assuming that $m \ll M$ and using the quadrupole formula for the metric perturbation associated with gravitational waves

(a) **[7 Marks]** Show that all the amplitudes $h_{\alpha\beta}$ of gravitational wave, emitted by such system, are periodic functions of time with $\omega = 2\omega_0$, where $\omega_0 = 2\pi/T$, and T is the orbital period;

SOLUTION B3(a) [seen similar]

$$x_1 = r\cos\omega_0 t,$$
$$x_2 = r\sin\omega_0 t,$$
$$D_{11} = mr_c^2(3\cos^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 + 3\cos 2\omega_0 t),$$
$$D_{22} = mr_c^2(3\sin^2\omega_0 t - 1) = \frac{1}{2}mr^2(1 - 3\cos 2\omega_0 t),$$
$$D_{12} = \frac{3}{2}mr_c^2\sin 2\omega_0 t,$$

then

$$h_{11} = -\frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\cos 2\omega_0,$$
$$h_{22} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\cos 2\omega_0 t = -\frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$
$$h_{12} = \frac{2Gmr^2}{3c^4R}\frac{3}{2}(2\omega_0)^2\sin 2\omega_0 t = \frac{4\omega_0^2 Gmr^2}{c^4R}\sin 2\omega_0,$$

it is clear, that

$$\omega = 2\omega_0.$$

(b) **[9 Marks]** Show that, to an order of magnitude (omitting the indices α and β)

$$h \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3},$$

where r_g is the gravitational radius of the mass m and R_g is the gravitational radius of the black hole.

SOLUTION B3(b) [unseen]

From

$$r\omega_0^2 = \frac{GM}{r^2},$$

we have

$$\frac{1}{r^3} = \frac{\omega_0^2}{GM},$$

and finally

$$r_c^{-1} = (4GM)^{-1/3}\omega^{2/3}.$$

Thus

$$h \approx \frac{4\omega_0^2 Gmr^2}{c^4 R} = \frac{r_g R_g}{rR} \approx \frac{r_g}{R}\left(\frac{R_g\omega}{c}\right)^{2/3}.$$

(c) [**6 Marks**] The future LISA mission will be able to detect gravitational waves with $h > 10^{-23}$, if $10^{-4}Hz < \omega < 3 \cdot 10^{-3}Hz$. From what distance will it be possible to detect gravitational radiation from the binary system, containing the black hole of mass $m = 3M_\odot$, moving along a circular orbit with radius $r = 10^4 R_g$ around the massive black hole of mass $M = 10^3 M_\odot$?

SOLUTION B3(c) [unseen]

$$\omega_0^2 = \frac{GM}{r^3} = \frac{c^2}{2}\frac{2GM}{c^2 r^3} = c^2\frac{R_g}{2r^3},$$

hence,

$$\omega_0 = c\sqrt{\frac{R_g}{2r^3}} = c\sqrt{\frac{R_g}{2\cdot 10^{12}R_g^3}} = \frac{10^{-6}c}{\sqrt{2}R_g} = \frac{10^{-4}Hz}{\sqrt{2}},$$

thus

$$\omega = 2\omega_0 = \sqrt{2}10^{-4}Hz \geq 10^{-4}Hz,$$

which means that the radiation is within LISA frequency range.

$$h = \frac{3\cdot 10^5}{3\cdot 10^{18}}\left(\frac{3\cdot 10^5\cdot 10^{-4}}{3\cdot 10^{10}}\right)^{2/3}(\frac{m}{M.})(\frac{R}{1pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}$$

$$\approx 10^{-19}(\frac{m}{M.})(\frac{R}{1\ pc})^{-1}(\frac{M}{M.})^{2/3}(\frac{\omega}{10^{-4}Hz})^{2/3}.$$

Then

$$h = \frac{3\cdot 10^5 cm}{R}(\frac{3\cdot 10^5\cdot 10^3\cdot 1.4\cdot 10^{-4}s^{-1}cm}{3\cdot 10^{10}})^{2/3} > 10^{-23},$$

if

$$R < 3\cdot 10^{23}\cdot 10^5 cm\cdot 10^{-4} \approx 1Mpc.$$

4. (a) [**8 Marks**] Derive the geodesic deviation equation

$$\frac{D^2\eta^i}{ds^2} = R^i_{klm} u^k u^l \eta^m,$$

where η^i is the 4-vector joining points on two infinitesimally close geodesics, and u^k is the 4-velocity along the geodesic.

SOLUTION B4(a) [book work]

$$\eta^i = \frac{\partial x^i}{\partial v} \quad \delta v \equiv v^i \delta v \quad \frac{\partial u^i}{\partial v} = \frac{\partial v^i}{\partial s} \quad u^i = \frac{\partial x^i}{\partial s}$$

$$u^i_{;\,k} v^k = v^i_{;\,k} u^k$$

$$\frac{D^2 v^i}{ds^2} = (v^i_{;\,k} u^k)_{;l} \;\; u^l = (u^i_{;\,k} v^k)_{;} u^l = u^i_{;\,k;\,l} v^k u^l + u^i_{;\,k} v^k_{;\,l} u^l$$

$$\frac{D^2 v}{ds^2} = (u^i_{;\,l} u^l)_{;k} + u^m R^i_{mkl} u^k v^l, \quad u^i_{;\,l} u^l = 0$$

(b) [**9 Marks**] Consider two neighboring particles freely falling from rest in the Schwarzschild gravitational field in the same radial direction. Using the geodesic deviation equation show that the component of the Riemann tensor which is responsible for the tidal force in the radial direction is

$$R^1_{001} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

SOLUTION B4(b)[unseen]

Since two particles move in the same radial directions their spatial coordinates are r, θ, ϕ and $r + \Delta r, \theta, \phi$ respectively. Hence the separation vector $\eta^i = \delta^i_1 \Delta r(t)$. The fact that these two particles are in rest means that four- velocity of each particle is $u^i = \delta^i_0$,
hence, as follows from geodesic deviation equation

$$\frac{D^2\eta^1}{ds^2} = R^i_{mkl} u^k u^m \eta^l = R^i_{mkl} \delta^k_0 \delta^m_0 \delta^l_1 \Delta r = R^1_{001} \Delta r.$$

$$R^1_{001} = \Gamma^1_{01,0} - \Gamma^1_{00,1} + \Gamma^1_{n0}\Gamma^n_{01} - \Gamma^1_{n1}\Gamma^n_{00}.$$
$$\Gamma^1_{01,0} = 0,$$

$$\Gamma^n_{01} = \frac{1}{2} g^{nm}\left(g_{0m,1} + g_{1m,0} - g_{01,m}\right) = \frac{1}{2}\delta^n_0 g^{00} g_{00,1},$$

$$\Gamma^n_{00} = \frac{1}{2} g^{nm} \left(2g_{0m,0} + g_{1m,0} - g_{00,m}\right) = -\frac{1}{2}\delta^n_1 g^{11} g_{00,1},$$

hence

$$R^1_{001} = \frac{1}{2}\left(g^{11} g_{00,1}\right)_{,1} - \frac{1}{4} g^{00} g^{11} \left(g_{00,1}\right)^2 + \frac{1}{2} g^{11} g_{00,1} \Gamma^1_{11}.$$

$$\Gamma^1_{11} = \frac{1}{2} g^{11} g_{11,1}.$$

Taking into account that

$$f = g_{00} = 1 - \frac{r_g}{r}, \quad g_{00,1} = \frac{r_g}{r^2}, \quad g_{00,1,1} = -\frac{2r_g}{r^3}$$

and

$$g^{00} = -g_{11} = \frac{1}{g_{00}}, \quad g^{11} = g_{00},$$

we have

$$R^1_{001} = \frac{1}{2}(-ff')' + \frac{1}{4f} ff'^2 + \frac{1}{4}(-f)^2 f'(-\frac{1}{f})' = -\frac{1}{2}(ff'' + ff'^2 - f'^2) =$$

$$= -\frac{1}{2}\left[-2(1 - \frac{r_g}{r})\frac{r_g}{r^3} + (1 - \frac{r_g}{r} - 1)\frac{r_g^2}{r^4}\right] =$$

$$= \frac{r_g}{r^3}\left(1 - \frac{r_g}{r}\right) + \frac{r_g^3}{2r^5} = \frac{r_g}{r^3}\left(1 - \frac{r_g}{r} + \frac{r_g^2}{2r^2}\right).$$

(c) **[5 Marks]** If the height of an observer is $l \approx 2$m, find the radial distance $r \gg r_g$ from a solar mass neutron star at which the radial tidal 3-acceleration experienced by the observer at rest ($a = c^2 \frac{D^2\eta^1}{ds^2}$) is equal to 100 $g \approx 10^3$ms^{-2}. You may assume that the observer's body is aligned along the radial direction and you may take the gravitational radius of the Sun to be 3 km.

SOLUTION B4(c) [unseen]

If $r \gg r_g$ we have

$$a \approx \frac{c^2 r_g l}{r^3} \approx 10^3 \text{m s}^{-2},$$

hence

$$r \approx 0.1 \left(c^2 r_g \text{s}^2\text{m}^{-1}\right)^{1/3} \approx 10^{-1}\left[\left(3 \cdot 10^8 \text{m s}^{-1}\right)^2 \cdot 3 \cdot 10^3 \text{ m} \cdot 2 \text{ m m}^{-1}; \text{s}^2\right]^{1/3} \approx$$

$$\approx 3 \cdot 10^{-1}\left(2 \cdot 10^19\right)^{1/3} \text{ m} \approx$$

$$\approx 800 \text{ km}.$$