Spacetime and Gravity: Assignment 6 Solutions

November 30, 2009

In what follows, unless otherwise stated, we will use units such that the speed of light, $c = 1$.

1.

We are given the Schwarzchild metric:

$$ds^2 = -\left(1 - \frac{2Gm}{r}\right) dt^2 + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2(\theta)d\phi^2) \quad (1)$$

where

$$r^2(d\theta^2 + \sin^2(\theta)d\phi^2) = r^2 d\Lambda^2_{(2)} \quad (2)$$

and we wish to remove the apparent coordinate singularity by transforming to Eddington-Finkelstein coordinates, in which:

$$\bar{t} = t + 2Gm \ln{(r - 2Gm)}. \quad (3)$$

We perform the substitution:

$$d\bar{t} = dt + 2Gm(r - 2Gm)^{-1} dr \quad (4)$$

and so

$$d\bar{t}^2 = dt^2 + 4Gm(r - 2Gm)^{-1} dr dt + 4(Gm)^2(r - 2Gm)^{-2} dr^2 \quad (5)$$

Substituting these back in our action gives:

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2Gm}{r}\right)(d\bar{t}^2 - 4Gm(r - 2Gm)^{-1} dr dt - 4(Gm)^2(r - 2Gm)^{-2} dr^2) \\
&\quad + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\Lambda^2_{(2)} \\
&= -\left(1 - \frac{2Gm}{r}\right) d\bar{t}^2 + \frac{4Gm}{r} dr dt + \frac{4(Gm)^2}{r^2}\left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 \\
&\quad + \left(1 - \frac{2Gm}{r}\right)^{-1} dr^2 + r^2 d\Lambda^2_{(2)} \\
&= -\left(1 - \frac{2Gm}{r}\right) d\bar{t}^2 + \frac{4Gm}{r} dr(d\bar{t} - 2Gm(r - 2Gm)^{-1} dr)
\end{aligned}$$

$$+\left(\frac{4(Gm)^2}{r^2}\left(1-\frac{2Gm}{r}\right)^{-1}+\left(1-\frac{2Gm}{r}\right)^{-1}\right)dr^2+r^2d\Lambda^2_{(2)}$$

$$= \quad -\left(1-\frac{2Gm}{r}\right)d\bar{t}^2+\frac{4Gm}{r}drd\bar{t}+\left(1-\frac{2Gm}{r}\right)^{-1}\left(1-\frac{4G^2m^2}{r^2}\right)dr^2+r^2d\Lambda^2_{(2)}$$

But

$$\begin{aligned}\left(1-\frac{2Gm}{r}\right)^{-1}\left(1-\frac{4G^2m^2}{r^2}\right)dr^2 &= \left(1-\frac{2Gm}{r}\right)^{-1}\left(1-\frac{2Gm}{r}\right)\left(1+\frac{2Gm}{r}\right)dr^2 \\ &= \left(1+\frac{2Gm}{r}\right)dr^2\end{aligned}$$

and so

$$ds^2=-\left(1-\frac{2Gm}{r}\right)d\bar{t}^2+\frac{4Gm}{r}drd\bar{t}+\left(1+\frac{2Gm}{r}\right)dr^2+r^2d\Lambda^2_{(2)} \tag{6}$$

which is the action we are looking for. Note that this has no singularity at the horizon, showing that the singularity appearing in the Schwartzchild metric is purely a coordinate one, not a physical one. The singularity appears from our choice of language to describe the physics, not from the physics itself.

2.

We are given the FRW equations for the scale factor $R(t)$:

$$\frac{3\dot{R}^2}{R^2}=8\pi\rho \qquad 2\frac{\ddot{R}R+\frac{1}{2}\dot{R}^2}{R^2}=-8\pi p \tag{7}$$

We are also told that the universe in this epoch is expanding as:

$$R(t)=R_0t^{\frac{1}{6}} \tag{8}$$

To find the effective equation of state of the matter in the universe during this epoch we start by solving the FRW equations for a general equation of state of the form

$$p=w\rho \tag{9}$$

So, dividing one equation by the other we obtain:

$$\frac{2(\ddot{R}R+\frac{1}{2}\dot{R}^2)}{3\dot{R}^2}=-w \tag{10}$$

To solve this for the equation of state we take an ansatz for $R(t)$ of the form:

$$R(t)=R_0t^y \tag{11}$$

where R_0 is a constant. Using this one obtains:

$$\ddot{R}R = R_0^2y(y-1)t^{2y-2} \tag{12}$$
$$\dot{R}^2 = R_0^2y^2t^{2y-2} \tag{13}$$

substituting these relations back into our original equation gives:

$$\begin{aligned} \frac{2(y(y-1)+\frac{1}{2}y^2)}{3y^2} &= -w && (14)\\ \Rightarrow \frac{-3y^2(1+w)}{2} &= -y && (15)\\ \Rightarrow y(\frac{3}{2}(1+w)y-1) &= 0 && (16) \end{aligned}$$

So either $y = 0$ or $y = \frac{2}{3(1+w)}$. Now we can match the w needed for the correct rate of expansion of the universe, i.e. $t^{\frac{1}{6}}$. So, set $y = \frac{1}{6}$ and:

$$\begin{aligned} y &= \frac{2}{3(1+w)} && (17)\\ \Rightarrow (1+w) &= 4 && (18)\\ \Rightarrow w &= 3 && (19) \end{aligned}$$

So for this rate of expansion of the universe $w = 3$, and the equation of state becomes $p = 3\rho$.

1 Summary of important concepts

1.It is important to distinguish between coordinate and physical singularities. Coordinate ones are those that can be removed via coordinate transformations (e.g the horizon singularity), physical ones are those that whatever coordinate transformation you do the singularity will still remain (e.g $r = 0$ in Schwartzchild).