University College London

Department of Physics and Astronomy

2B21 Mathematical Methods in Physics & Astronomy

Suggested Solutions for Problem Sheet M5 (2003–2004)

1. Rewrite the equation in the form

$$\frac{dy}{y} = 2\frac{x^3}{1+x^2} = 2x - 2\frac{x}{1+x^2}\ ,$$

which can be integrated to give

$$\ell n(y) = x^2 - \ell n(1+x^2) + C\ . \qquad \textbf{[2]}$$

The boundary condition that $y = 1$ when $x = 0$ means that the integration constant $C = 0$, and so the solution is

$$y = (1+x^2)^{-1}\, e^{x^2}\ . \qquad \textbf{[2]}$$

Look now for a solution of the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\ , \\ y' &= \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1}\ , \end{aligned}$$

with $a_0 \neq 0$. Inserting this into

$$(1+x^2)\,\frac{dy}{dx} = 2x^3 y\ ,$$

we find

$$\sum_{n=0}^{\infty} a_n(n+k)\, x^{n+k-1} + \sum_{n=0}^{\infty} a_n(n+k)\, x^{n+k+1} = 2\sum_{n=0}^{\infty} a_n\, x^{n+k+3}\ . \qquad \textbf{[1]}$$

The lowest power of x comes from the first term with $n = 0$. Hence $a_0 k = 0$ but, since $a_0 \neq 0$, the indicial equation gives $k = 0$ as the unique solution. **[1]** Therefore

$$\sum_{n=1}^{\infty} a_n n\, x^{n-1} + \sum_{n=1}^{\infty} a_n n\, x^{n+1} = 2\sum_{n=0}^{\infty} a_n\, x^{n+3}\ .$$

The only x^0 term only exists in the first sum, which means that $a_1 = 0$ and in general all the odd coefficients vanish. There is an x^1 term also only in the first sum so the coefficient $a_2 = 0$ as well. **[2]**

Now change the dummy index n so that one sees the same power of x in all three sums:

$$\sum_{n=-3}^{\infty} a_{n+4}(n+4)\, x^{n+3} + \sum_{n=-1}^{\infty} a_{n+2}(n+2)\, x^{n+3} = 2\sum_{n=0}^{\infty} a_n\, x^{n+3}\ ,$$

which leads to the recurrence relation

$$(n+4)a_{n+4} + (n+2)a_{n+2} = 2a_n \ . \qquad \textbf{[2]}$$

We have proved that $a_2 = 0$ and, since $y = 1$ when $x = 0$, we know that $a_0 = 1$. Putting $n = 0$ into the recurrence relation, we find that $a_4 = \frac{1}{2}$ and so $y \approx 1 + \frac{1}{2}x^4 + O(x^6)$. **[2]**

Expanding the two factors in the exact solution as power series,

$$y \approx \left(1 - x^2 + x^4 + O(x^6)\right) \left(1 + x^2 + \tfrac{1}{2}x^4 + O(x^6)\right) \approx 1 + \tfrac{1}{2}x^4 + O(x^6) \ , \qquad \textbf{[2]}$$

which agrees with the earlier result.

2. Look for a solution of the second order differential equation

$$(2x + x^2)\,\frac{d^2y}{dx^2} + (1+x)\,\frac{dy}{dx} - p^2 y = 0$$

in the form

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n\, x^{n+k}\,, \\ y' &= \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1}\,, \\ y'' &= \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)\, x^{n+k-2}\,. \end{aligned} \qquad \textbf{[1]}$$

Inserting these into the equation, we obtain

$$\sum_{n=0}^{\infty} 2a_n (n+k)(n+k-1)\, x^{n+k-1} + \sum_{n=0}^{\infty} a_n (n+k)(n+k-1)\, x^{n+k}$$

$$+ \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k-1} + \sum_{n=0}^{\infty} a_n (n+k)\, x^{n+k} - p^2 \sum_{n=0}^{\infty} a_n\, x^{n+k} = 0\,. \qquad \textbf{[1]}$$

Grouping like powers together, this simplifies to

$$\sum_{n=0}^{\infty} a_n\,(n+k)(2n+2k-1)\, x^{n+k-1} + \sum_{n=0}^{\infty} a_n\, \left[(n+k)^2 - p^2\right]\, x^{n+k} = 0\,. \qquad \textbf{[2]}$$

If this is to be true for a range of values of x, it must be true power by power in x. The lowest power comes from $n = 0$ in the first term. Since there is no x^{k-1} power in the second term, we demand that

$$a_0\, k(2k-1) = 0\,.$$

However, by definition, $a_0 \neq 0$ so that $k = 0$ or $k = \frac{1}{2}$. **[2]**

To get the recurrence relation, change the dummy index so that we have the same powers of x everywhere by putting $n \to n+1$ in the first term:

$$\sum_{n=-1}^{\infty} a_{n+1}\,(n+k+1)(2n+2k+1)\, x^{n+k} + \sum_{n=0}^{\infty} a_n\, \left[(n+k)^2 - p^2\right]\, x^{n+k} = 0\,. \qquad \textbf{[2]}$$

This gives us immediately the recurrence relation:

$$\frac{a_{n+1}}{a_n} = -\frac{(n+k)^2 - p^2}{(n+k+1)(2n+2k+1)} \,,$$

with $k = 0$ or $k = \frac{1}{2}$. **[2]**

The series converges if, when $n \to \infty$,

$$\left|\frac{a_{n+1}x^{n+1}}{a_n x^n}\right| = \left|\frac{a_{n+1}}{a_n}\right| \, |x| \; < \; 1 \,. \qquad \textbf{[1]}$$

This means that

$$\left|\frac{(n+k)^2 - p^2}{(n+k+1)(2n+2k+1)}\right| \, |x| \to \tfrac{1}{2}\,|x| < 1 \,,$$

i.e. $|x| < 2$. **[2]**

On the other hand, if p is a positive integer the recurrence relation tells us for the $k = 0$ solution that

$$\frac{a_{p+1}}{a_p} = -\frac{(p^2 - p^2)}{(p+1)(2p+1)} = 0 \,. \qquad \textbf{[2]}$$

Since there are only two terms in the recurrence relation, all subsequent a_n vanish and the series terminates to give the polynomial $T_p(x)$. **[1]**

Given that $T_p(0) = 1$, i.e. $a_0 = 1$, the recurrence relation leads to $a_1 = p^2\,a_0 = p^2$ so that, to order x, the $k = 0$ solution is

$$T_p(x) \approx 1 + p^2\,x \,. \qquad \textbf{[2]}$$

Therefore

$$2T_p(x)\,T_q(x) \approx 2(1 + p^2 x)(1 + q^2 x) \approx 2 + 2(p^2 + q^2)x \,. \qquad \textbf{[1]}$$

Looking at the other side,

$$T_{p+q}(x) + T_{p-q}(x) \approx 1 + (p+q)^2 x + 1 + (p-q)^2 x$$

$$= 2 + (p^2 + 2pq + q^2)x + (p^2 - 2pq + q^2)x = 2 + 2(p^2 + q^2)x \,. \qquad \textbf{[1]}$$

Thus the identity is satisfied at least to first order in x.