UNIVERSITY COLLEGE LONDON
DEPARTMENT OF PHYSICS AND ASTRONOMY

2B21 MATHEMATICAL METHODS IN PHYSICS AND ASTRONOMY

Problem Sheet M3 (2003–2004)

Solutions to be handed in on Tuesday 28 October 2003

1. My favourite result in matrix theory is the following expression for the determinant of the matrix $\underline{I}+\varepsilon\,\underline{A}$:

$$|\underline{I}+\varepsilon\,\underline{A}| = \exp\left[tr\left\{\ell n\left(\underline{I}+\varepsilon\,\underline{A}\right)\right\}\right] .$$

Here $\underline{A}$ is any square matrix, $\underline{I}$ the corresponding unit matrix, and ε is a small number. The logarithm is defined by its series expansion in powers of ε with $\ell n(\underline{I}) = \underline{0}$. The trace of a square matrix $\underline{B}$, $\mathrm{tr}\,(\underline{B})$, is the sum of its diagonal elements.

Show that to second order in ε

$$|\underline{I}+\varepsilon\,\underline{A}| = 1+\varepsilon\,tr\,(\underline{A}) + \tfrac{1}{2}\varepsilon^2\left[(tr\,\underline{A})^2 - tr\,(\underline{A}^2)\right] + 0(\varepsilon^3)\,. \qquad \textbf{[6 marks]}$$

Verify the theorem for the matrix $\underline{A} = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$. **[4 marks]**

2. Find the eigenvalues of the matrix $\underline{A} = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}$. **[2 marks]**

Show that $\underline{A}^2 = \underline{I} + 2\underline{A}$ and <u>hence</u> evaluate $\underline{A}^4$ and $\underline{A}^8$. **[4 marks]**

If t_n is defined in terms of the trace of a matrix through

$$t_n = \left[tr(\underline{A}^n)\right]^{1/n} ,$$

show that $t_2 \approx 2.4495$, $t_4 \approx 2.4147$, and $t_8 \approx 2.4142$. **[3 marks]**

Why does $t_n \to \sqrt{2}+1$ as $n\to\infty$? **[3 marks]**

3. Given that $\underline{A}$ is an anti-Hermitian matrix, $\underline{A}^\dagger = -\underline{A}$, show from <u>first</u> principles that its eigenvalues are either purely imaginary or zero. **[5 marks]**
Verify this result for the following matrix, where one eigenvalue is $\lambda = i$,

$$\underline{A} = \begin{pmatrix} 0 & 1+i & i \\ -1+i & 0 & 1-i \\ i & -1-i & 0 \end{pmatrix} . \qquad \textbf{[5 marks]}$$