Lecture 5 Radial oscillations of stars

In this lecture we consider oscillations of a star about its spherically symmetric equilibrium state, with the oscillatory motion being purely radial.This is relevant to a number of classical variable stars, *e.g.* Cepheids. A readable account of the theory of radial stellar pulsations may be found in [5.1], while [5.2] provides a more mathematical treatment.

5.1 Linear adiabatic wave equations for radial oscillations

We shall assume that the amplitude of the oscillations is small, so that linear perturbation theory will suffice, and in the present section we shall suppose that the period of the oscillations is sufficiently short that no heat is exchanged between neighbouring fluid elements, *i.e.* the oscillations are adiabatic. Then the linearized equations for small perturbations are equations (2.19, 2.21):

$$\rho_0 \frac{\partial \mathbf{u}}{\partial t} = -\nabla p' - \rho' \nabla \psi_0 - \rho_0 \nabla \psi', \qquad (5.1a)$$

$$\frac{\partial \rho'}{\partial t} = -\nabla \cdot (\rho_0 \mathbf{u}), \qquad (5.1b)$$

$$\nabla^2 \psi' = 4\pi G \rho', \qquad (5.1c)$$

$$\frac{\partial p'}{\partial t} + \mathbf{u} \cdot \nabla p_0 = \frac{\Gamma_1 p_0}{\rho_0}\left(\frac{\partial \rho'}{\partial t} + \mathbf{u} \cdot \nabla \rho_0\right). \qquad (5.1d)$$

We now seek solutions with sinusoidal time dependence. Writing

$$\begin{aligned} \delta \mathbf{r} &= \hat{\mathbf{r}} U(r) \exp(i\omega t), \\ p' &= p_1(r) \exp(i\omega t), \\ \rho' &= \rho_1(r) \exp(i\omega t), \qquad (5.2) \\ \psi' &= \psi_1(r) \exp(i\omega t), \end{aligned}$$

and $\mathbf{u} = \partial \delta \mathbf{r} / \partial t = i\omega \delta \mathbf{r}$, equations (5.1) become

$$-\rho_0\omega^2 U = -\frac{d}{dr}p_1 - \rho_1 g_0 - \rho_0 \frac{d}{dr}\psi_1, \qquad (5.3a)$$

$$\rho_1 = -\frac{1}{r^2}\frac{d}{dr}\left(r^2\rho_0 U\right), \qquad (5.3b)$$

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\psi_1\right) = 4\pi G\rho_1, \qquad (5.3c)$$

$$p_1 - \rho_0 g_0 U = c^2\left(\rho_1 + \frac{d\rho_0}{dr}U\right), \qquad (5.3d)$$

by using $d\psi_0/dr = g_0,\ \ dp_0/dr = -\rho_0 g_0$, and $\Gamma_1 p_0/\rho_0 = c^2$,

where c is the adiabatic sound speed. Eliminating ρ_1 and ψ_1 in equations (5.3), we arrive to

$$\frac{dU}{dr} = \left(\frac{g_0}{c^2} - \frac{2}{r}\right)U - \frac{1}{\rho_0 c^2}p_1, \qquad (5.4a)$$

$$\frac{dp_1}{dr} = \left(\omega^2 - N^2 + 4\pi G\rho_0\right)\rho_0 U - \frac{g_0}{c^2}p_1, \qquad (5.4b)$$

where N is the so-called Brunt-V�is�l� frequency,

$$N^2 = -g_0\left(\frac{d\ln\rho_0}{dr} - \frac{1}{\Gamma_1}\frac{d\ln p_0}{dr}\right). \qquad (5.5)$$

Exercise 5.1. Convince yourself, using the divergence theorem, that

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 u\right)$$

for any spherically-symmetric vector field $\mathbf{u} = u(r)\hat{\mathbf{r}}$. Further, fill in the missing steps to derive equations (5.4). [Hint: from equations (5.3b,c) we have $d\psi_1 / dr = -4\pi G\rho_0 U$].

5.2 Boundary conditions

Equations (5.4) represent a system of two ordinary differential equations for $U(r)$ and $p_1(r)$. In order to solve this system, we require boundary conditions at the center of the star ($r = 0$) and at its surface ($r = R$).

The boundary condition at the center is that the solution be regular, not divergent, at the origin. We are looking for the solutions $U(r)$ in the vicinity of $r = 0$ as a power series expansion

$$U(r) = a_0 r^{\alpha} + a_1 r^{\alpha+1} + \dots \qquad (5.6)$$

The first term in the series specifies the leading-order behaviour of $U(r)$ near the origin; we require $a_0 \neq 0$ and allow α to be an unknown constant which has to be determined. With this expansion for $U(r)$, equation (5.4a) provides a corresponding expansion for $p_1(r)$:

$$p_1(r) = -(\alpha+2)\rho_0(0)c^2(0)a_0 r^{\alpha-1} + \dots , \qquad (5.7)$$

where $\rho_0(0)$ and $c(0)$ are the central values of the equilibrium density and the sound speed, and we limit our analysis by the leading-order term only.

We now substitute the expansions (5.6, 5.7) into the second equation (5.4b):

$$-(\alpha-1)(\alpha+2)\rho_0(0)c^2(0)a_0 r^{\alpha-2} + \dots = 0. \qquad (5.8)$$

As before, dots designate terms of higher order, proportional to $r^{\alpha-1}$, r^{α} etc. From equation (5.8), we get $\alpha = 1$ or $\alpha = -2$. The requirement that the solution be regular rules out $\alpha = -2$; hence we must have $\alpha = 1$. The regularity condition thus selects one of the two linearly-independent solutions of the oscillation equations, which behaves as

$$U(r) \simeq a_0 r, \quad p_1(r) \simeq -3\rho_0(0)c^2(0)a_0 \qquad (5.9)$$

in the vicinity of $r = 0$. Constant a_0 is arbitrary, since the equations are homogeneous (solutions are only determined with an arbitrary scaling factor).

At the surface, we require the Lagrangian pressure perturbation δp to be zero. With $\delta p = p' + \delta \mathbf{r} \cdot \nabla p_0$ (equation 2.25 of Lecture 2), the outer boundary condition is

$$p_1 - \rho_0 g_0 U = 0, \qquad r = R. \qquad (5.10)$$

In a realistic stellar model, we do not have any well-defined "surface", but rather a smooth transition to the low-density atmosphere. the boundary condition $\delta p = 0$ is still applicable, however, when the "stellar radius" is taken to be sufficiently high in the atmosphere. The physical reason is that the atmospheric layers above $r = R$ have essentially no dynamical influence on the global oscillations due to their very small mass. Also when $\rho_0(R)$ is small, the boundary condition (5.10) can be replaced with $p_1(R) = 0$.

5.3 Eigenvalue nature of the problem

The second-order system of differential equations (5.4) has two linearly-independent solutions. Regularity condition at $r = 0$ selects one of them; this solution does not, in general, satisfy the surface boundary condition (5.10). The second boundary condition can be satisfied for certain values of ω^2 only: these values are called eigenvalues, and corresponding solutions for $U(r)$ and $p_1(r)$ are called eigenfunctions. The eigenvalues give the resonant frequencies (eigenfrequencies) at which the star can oscillate radially.

5.4 Local dispersion relation and mode classification

Let us suppose for a moment that we have some solutions $U(r)$ and $p_1(r)$ to the differential equations (5.4) with a sinusoidal character, which oscillate rapidly with r. We will see later in this section that this is indeed the case in the high-frequency limit (when ω is high). We write these solutions as

$$U(r) = \bar{U}(r)\exp(ikr), \quad p_1(r) = \bar{p}_1(r)\exp(ikr) \quad (5.11)$$

with slowly-varying amplitude functions $\bar{U}(r)$ and $\bar{p}_1(r)$, and rapidly-varying exponent (k is high). Differentiation with respect to r gives

$$\frac{dU}{dr} = ikU, \qquad \frac{dp_1}{dr} = ikp_1, \qquad (5.12)$$

when k is large (contribution of terms with derivatives of the amplitude functions can be neglected). We substitute (5.12) into the oscillation equations (5.4), getting

$$ikU + \frac{1}{\rho_0 c^2} p_1 = 0, \qquad (5.13a)$$

$$\left(\omega^2 - N^2 + 4\pi G\rho_0\right)\rho_0 U - ikp_1 = 0. \qquad (5.13b)$$

Note that we were using the asymptotic limit of large k once more, neglecting g_0 / c^2 and $2 / r$ compared with k.

Equations (5.13) are algebraic equations. This homogeneous system of two linear equations for U and p_1 has a non-trivial solution if and only if the determinant of matrix of its coefficients iz zero, *i.e.* when

$$k^2 = \frac{\omega^2}{c^2}\left(1 - \frac{N^2}{\omega^2} + \frac{4\pi G\rho_0}{\omega^2}\right). \qquad (5.14)$$

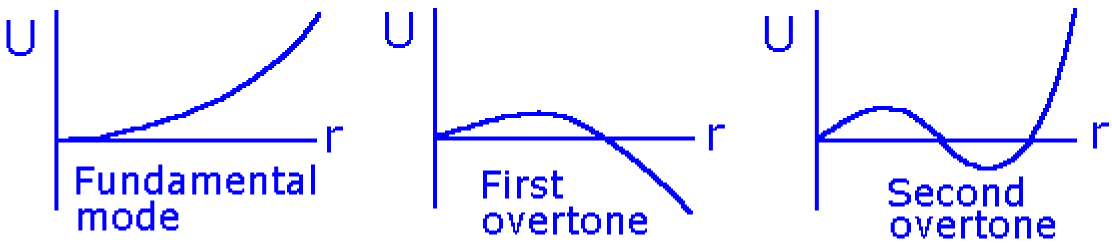
We observe from this relation that k^2 is large when ω^2 is large, so that our local analysis is indeed applicable in the asymptotic limit of high frequencies.

What we also observe is that when ω is high, k^2 is approximately ω^2 / c^2, which is nothing else but the dispersion relation for sound waves developed earlier in Lecture 2 (equation 2.32). We thus arrive to the simple physical interpretation of the radial oscillations: at least in the high-frequency limit, they are formed by the acoustic waves propagating along stellar radius. A wave travelling upwards is reflected back at the stellar surface, and the downward wave is reflected back at the stellar center. When added together, these waves can form a standing wave (a particular mode of stellar oscillations), which can only happen at frequencies of acoustic resonances (radial oscillation frequencies).

When going to smaller frequencies, simple local analysis can loose its accuracy. Note also that it is always in trouble in the vicinity of $r = 0$: indeed, we were neglecting

$2/r$ compared with k. By doing that we have the effects of spherical geometry discarded: we are using a plane-wave approximation to describe spherical waves.

When ω gets higher, the acoustic wavelength becomes smaller, and radial displacement function $U(r)$ acquires more and more modes in its variation with radius. Different modes of radial oscillations are classified according to the number of nodes in $U(r)$. The oscillation with no nodes, which has the lowest frequency, is called the fundamental radial mode. Higher in frequency is the first overtone, with one node in $U(r)$, and so on.

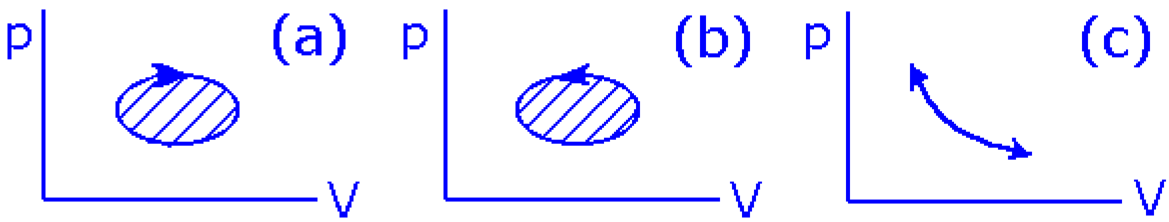


5.5 Non-adiabatic oscillations: physical discussion of driving and damping

Consider a small fluid element of volume V. Let it be small enough so that at any time, pressure p can be considered as uniform everywhere inside V. If V increases by an ammount dV, the mechanical work done by the fluid element against the external pressure is $p\,dV$. Integrating over one cycle of the oscillation, this work will be

$$\oint p\,dV.$$

If this work is positive, it goes to the increase of the total mechanical energy of the surrounding fluid, *i.e.* to the increase of of the pulsational energy of the star; the fluid element acts as a driving source (Fig.a):



In the driving case, at point of maximum compression (V minimum, $DV/Dt = 0$), p is still increasing, $Dp/Dt > 0$ (Fig.a), which means that some heat is being added. What we have is assentially a small heat engine, which transfers thermal energy into mechanical energy.

One way in which oscillations can be driven is by the so-called opacity mechanism, or κ- mechanism. It operates in the near-surface layers of partial ionization with anomalous behaviour of the opacity κ. If the opacity increases when the star is compressed, the heat is gained due to the blocking of the radiative flux coming from the stellar interior. Another driving mechanism is the so-called ε- mechanism. When the star is compressed, density and temperature increase, hence the nuclear energy generation rate ε increases and so more heat is generated.

If the work integral is negative (Fig.b), the fluid element acts as an energy sink for the oscillation, absorbing the mechanical energy from the surrounding fluid and transfering it into heat. At maximum compression, the heat is being lost. This scenario is realized in the so-called radiative damping, when the compressed (and hence hotter) fluid element looses its heat to the surrounding. Note that in the adiabatic approximation, we neglect any heat exchange, and the work integral is zero (Fig.c). Further discussion of the various excitation mechanisms may be found in [5.2].

LITERATURE

[5.1] Cox, J. P., 1980. *Theory of stellar pulsation* (Princeton University Press: Princeton)

[5.2] Unno, W., Osaki, Y., Ando, H., Saio, H. & Shibahashi, H., 1989. *Nonradial oscillations of stars* (2nd edition) (University of Tokyo Press)