

Lecture 4 Theory of rotating bodies

4.1 Equilibrium equations for a slowly rotating body

In this lecture we shall be considering principally how to calculate the shape of a fluid body that is rotating slowly with a uniform rotation rate. We shall consider in particular the case of a slowly rotating star; but the equations apply equally well to, for example, a slowly rotating gaseous planet. It will be assumed that in the absence of rotation the body would be spherically symmetric, and that rotation induces a weak distortion of the shape from spherical symmetry. For slow rotation, the distorted body is axisymmetric about the rotation axis, as one would expect. Although we shall not consider it here, faster rotation can give rise to some surprises, notably the Jacobi ellipsoids which are triaxial figures of equilibrium. For a fuller exposition of the subject, see the classic texts [4.1] and [4.2].

The momentum equation (1.9) is

$$\frac{D\mathbf{u}}{Dt} = -\frac{1}{\rho} \nabla p - \nabla \psi, \quad (4.1)$$

assuming that the only body forces are due to gravity. Since the rotation is axially symmetric, it is convenient to introduce cylindrical polar coordinates (r_c, φ, z) , z being the rotation axis; r_c is the distance from the rotation axis, and φ is the angular coordinate, so that the fluid velocity \mathbf{u} is directed along φ .

For a steady rotation, we have $\partial \mathbf{u} / \partial t = 0$, but $D\mathbf{u} / Dt$ differs from zero: the velocity of a given fluid element does not change with time in its absolute value, but changes in its direction. Simple geometrical considerations show that for an observer moving together with the fluid element,

$$d\mathbf{u} = -\hat{\mathbf{r}}_c u d\varphi, \quad (4.2)$$

where $\hat{\mathbf{r}}_c$ is unit vector along r_c and $u = |\mathbf{u}|$. Introducing the angular velocity of rotation $\Omega = D\varphi / Dt$, we have

$$\frac{D\mathbf{u}}{Dt} = -\hat{\mathbf{r}}_c u \Omega = -\hat{\mathbf{r}}_c r_c \Omega^2, \quad (4.3)$$

since $u = r_c \Omega$. Indeed, the equation (4.2) gives nothing else but the centripetal acceleration of the circular motion.

For a uniform rotation (Ω is constant in space), the right-hand side of the equation (4.3) can be written as a gradient of scalar function:


$$-\hat{\mathbf{r}}_c r_c \Omega^2 = -\nabla \left(\frac{1}{2} \Omega^2 r_c^2 \right), \quad (4.4)$$

and the momentum equation (4.1) reduces to

$$\nabla p = -\rho \nabla \Phi, \quad (4.5)$$

where

$$\Phi = \psi - \frac{1}{2} \Omega^2 r_c^2. \quad (4.6)$$

Exercise 4.1 Now, instead of uniform rotation, consider a general rotation law around the Z -axis. Show that the centripetal acceleration $-\hat{\mathbf{r}}_c r_c \Omega^2$ can be written as a gradient of a scalar function if and only if the rotation rate is independent of φ and z , i.e. $\Omega = \Omega(r_c)$ only. 

We can argue qualitatively from equation (4.4) what the effect of rotation on the equilibrium shape of the body will be. For an observer moving together with the fluid - in the corotating coordinate system, the fluid is at rest; instead of the centripetal force of the circular motion, in the corotating frame we have the fictitious centrifugal force (a body force), which has to be balanced by pressure and gravity. On the stellar surface, the centrifugal acceleration is zero, and it is radially outwards at the equator. It thus reduces the "effective gravitational acceleration" at the equator, i.e. the centreward pull is not so strong there as at the poles and so instead of being spherical the body "bulges" at the equator.

The gradient vector ∇f of any scalar is perpendicular to surfaces of constant f . Thus it follows from equation (4.5) that surfaces of constant p are also surfaces of constant Φ , and vice versa. Thus we can write $p = p(\Phi)$, and so

$$\nabla p = \frac{dp}{d\Phi} \nabla \Phi.$$

Substituting this into equation (4.5) yields

$$\rho = -\frac{dp}{d\Phi},$$

so ρ is also a function of Φ : $\rho = \rho(\Phi)$.

Henceforward, for definiteness, we shall speak of the body as being a star, but it could equally be a gaseous planet, for example. The outer surface of the star is a surface of constant pressure (because the pressure outside is constant, say zero) and so on the surface, Φ is constant.

We consider the case where if the star were not rotating it would be spherically symmetric, and rotation induces a *weak* distortion from sphericity. We suppose that the star has mass M and (in the nonrotating case) radius R .

Let us approximate the gravitational potential Ψ by what it would be in the nonrotating case:

$$\Psi = -\frac{GM}{r}$$

at the surface and outside the star.

We suppose that the surface of the rotating star is described by

$$r = R(1 + \varepsilon(\theta)), \quad (4.7)$$

where $\varepsilon(\theta)$ is a small function of the colatitude θ . Then, using $r_c = r \sin \theta$, we have on the surface

$$\Phi_{\text{surf}} = -\frac{GM}{R(1 + \varepsilon)} - \frac{1}{2}\Omega^2 R^2 (1 + \varepsilon)^2 \sin^2 \theta \quad (4.8)$$

constant (*i.e.* independent of θ). The rotation is slow and the distortion weak, so Ω^2 and ε are small and we neglect products of small quantities. Then (4.8) implies that

$$-\frac{GM}{R}(1 - \varepsilon) - \frac{1}{2}\Omega^2 R^2 \sin^2 \theta$$

is independent of θ , i.e.

$$\varepsilon(\theta) = \frac{1}{2} \frac{\Omega^2 R^3}{GM} \sin^2 \theta \quad (4.9)$$

(possibly plus a constant). Note that $\Omega^2 R$ is the equatorial acceleration due to centrifugal forces; and GM/R^2 is the gravitational acceleration. So the dimensionless quantity $\Omega^2 R^3 / GM$ is the ratio of centrifugal acceleration to gravitational acceleration.

The radius at the pole and at the equator are obtained from equation (4.7) by putting $\theta = 0$ and $\theta = \pi/2$ respectively. Thus the relative difference between equatorial and polar radii is

$$\frac{R(1 + \varepsilon(\pi/2)) - R(1 + \varepsilon(0))}{R} = \frac{1}{2} \frac{\Omega^2 R^3}{GM}. \quad (4.10)$$

The only thing wrong with this argument is the use of $\psi = -GM/r$. In general, we should properly use the gravitational potential appropriate to the distorted star. The relevant generalization of our analysis is known as the Chandrasekhar-Milne expansion. A description of the procedure is given in [4.3].

The distortion of the gravitational field can indeed modify our result (equation 4.10) significantly. The magnitude of this effect depends on the mass distribution inside the star: the distortion of the gravitational field is bigger when more mass is localized closer to the surface, where centrifugal forces are bigger. For a star with uniform density, the coefficient 1/2 in the right-hand side of (4.10) appears to be replaced with 5/4. The rotational distortion of the stellar configuration becomes bigger; it is indeed quite a general property of gravity to have a destabilizing effect. In the opposite limit, when almost all the mass is concentrated very near the stellar center, the equation (4.10) is adequate. For a real star, we can well expect the result to be in between the two limits.

4.2 Binary stars

Consider a binary system in which the two components are in circular orbits about their common centre of mass O , and in which the two stars corotate so as to always show the same side to the other star. In this system there is a rotating frame in which the stars are completely stationary. If Ω is the angular velocity of each star about O , in an inertial frame, then of course Ω is also the angular velocity of the rotating frame.

Suppose that the separation distance between the two stars is a , that their masses are M_1, M_2 , and that their respective distances from O are μa and $(1 - \mu)a$. Since O is the centre of mass,

$$\mu = \frac{M_2}{M_1 + M_2}. \quad (4.11)$$

Also the gravitational force on star 1 towards star 2 (and hence towards O) must be equal to $M_1 \Omega^2 (\mu a)$, since μa is the radius of its circular orbit; hence it is straightforward to show that

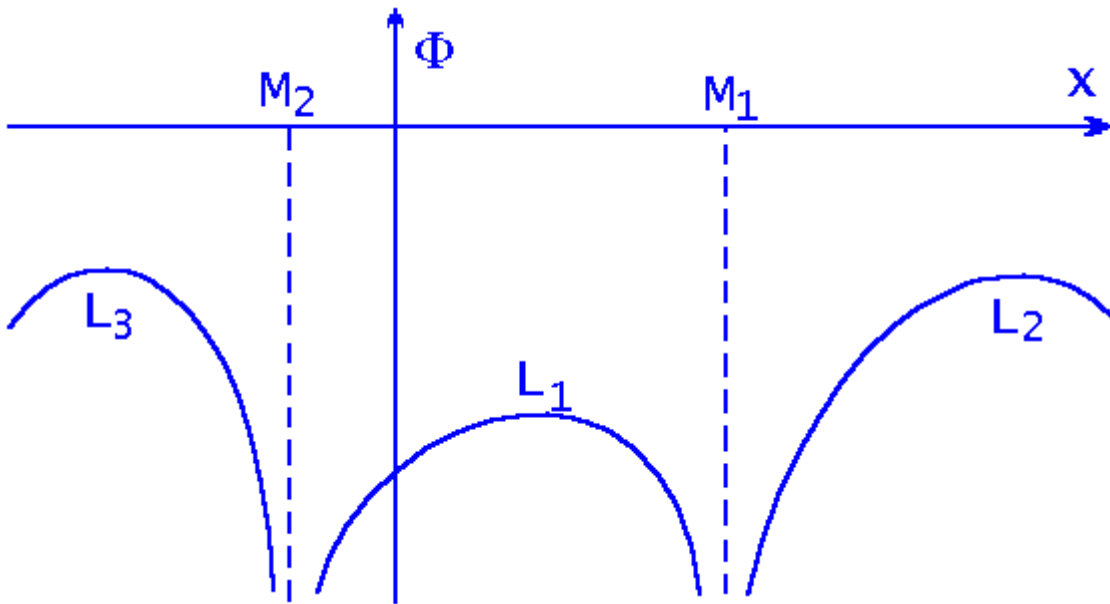
$$\Omega^2 = \frac{G(M_1 + M_2)}{a^3}. \quad (4.12)$$

Now equations (4.5) and (4.6) hold for this system (in the rotating frame), where the gravitational potential Ψ is given by the sum of the potentials due to the two stars. Choosing Cartesian coordinates such that the angular velocity of the frame is in the Z -direction, with the stars at $(\mu a, 0, 0)$ and $(-(1 - \mu)a, 0, 0)$, Φ can be written from equation (4.6) as

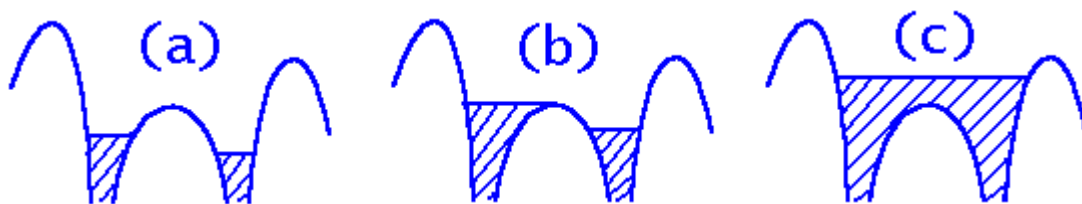
$$\Phi = -\frac{GM_1}{(x - \mu a)^2 + y^2 + z^2} - \frac{GM_2}{(x + (1 - \mu)a)^2 + y^2 + z^2} - \frac{1}{2} \Omega^2 (x^2 + y^2) \quad (4.13)$$

(the Roche potential). Here we have made the same approximation as in section 4.1, namely that we can use the undistorted gravitational potential of each star: this is reasonable for centrally condensed stars.

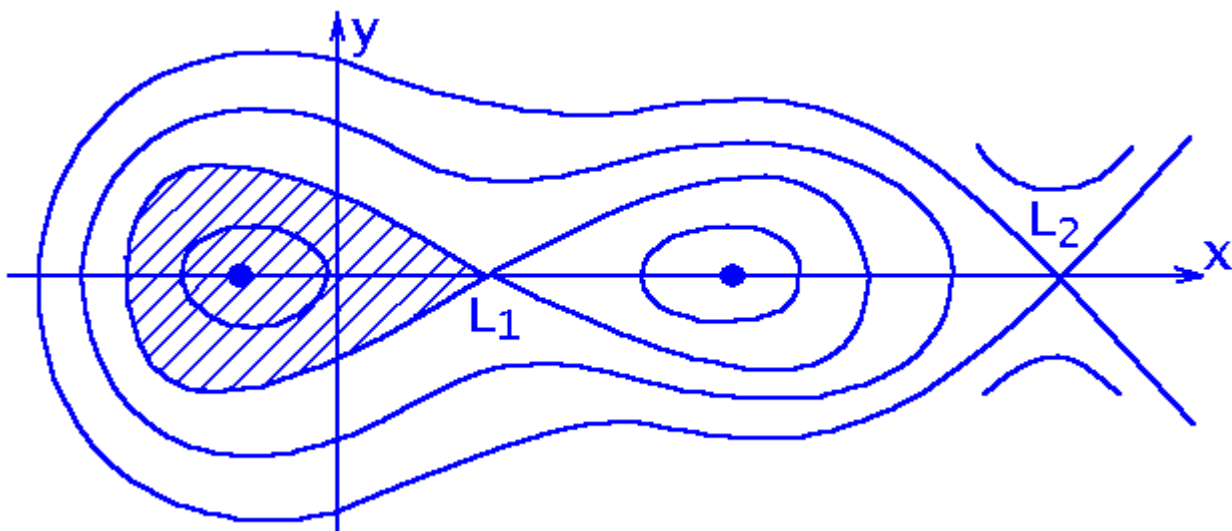
The potential (4.13) is plotted schematically as a function of x along the line $y = z = 0$ below. The Lagrangian points L_1, L_2 and L_3 , where $\nabla \Phi = 0$, are indicated.



As in the case of a single star, the surface of each star in the binary system is a surface of constant Φ . Now provided the surface potential of each star is less than the potential Φ_{L1} at the L_1 Lagrangian point, each star occupies a well in the Roche potential and the stars form a detached binary system (Fig. a):



Suppose though that M_2 expands (perhaps attempting to become a red giant) until its surface potential is equal to Φ_{L1} (Fig. b). Any further expansion will cause matter to fall from star 2 to star 1, since it will fall to the lower potential. Algol is an example of such a binary. Finally, if the surface potentials of both stars are greater than Φ_{L1} , then we have a "common envelope binary" or "contact binary" (Fig. c). It is also instructive to plot contours of constant Φ in the $x - y$ plane:



The shaded region, called a Roche lobe, is the maximum region the star can occupy before it starts to lose mass to its companion.

4.3 Dynamics of rotating stellar models

We have not so far considered how energy is transported in the rotating star. A well-known result, which is discussed at length in [4.3] is that one cannot have a uniformly rotating star in strict radiative equilibrium. Assuming the contrary leads to what is known as "von Zeipel's paradox". The same is true if the rotation rate is a function only of distance from the rotation axis. We conclude therefore that the rotation rate must have a more general form, depending on cylindrical polar coordinate Z as well as distance from the axis, or that strict radiative equilibrium does not hold.

The von Zeipel paradox in effect says that the radiative flux cannot be balanced everywhere by the energy generation. Some regions have a net influx of heat: these will heat up and tend to rise under buoyancy. Others will cool and sink. This tends to set up motions in meridional planes: this is called meridional circulation.

LITERATURE

[4.1] Chandrasekhar, S., 1969. *Ellipsoidal Figures of Equilibrium* (Yale University Press: New Haven)

[4.2] Lyttleton, R. A., 1953. *The stability of rotating liquid masses* (C.U.P.: Cambridge)

[4.3] Tassoul, J.-L., 1978. *Theory of rotating stars* (Princeton University Press: Princeton).